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Analysis and discretization of the volume penalized Laplace operator with Neumann boundary conditions

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ABSTRACT

We study the properties of an approximation of the Laplace operator with Neumann boundary conditions using volume penalization. For the one-dimensional Poisson equation we compute explicitly the exact solution of the penalized equation and quantify the penalization error. Numerical simulations using finite differences allow then to assess the discretization and penalization errors. The eigenvalue problem of the penalized Laplace operator with Neumann boundary conditions is also studied. As examples in two space dimensions, we consider a Poisson equation with Neumann boundary conditions in rectangular and circular domains.

1. Introduction

Solving partial differential equations (PDEs) in complex domains is unavoidable in real world applications. Different numerical methods have been developed so far, for example body fitted computational grids or coordinate transforms [4]. Immersed boundary methods are still of growing interest due to their high flexibility and their ease of implementation into existing codes. The underlying idea of these methods is to embed the complex geometry into a simple geometry (*e.g.* a rectangle) for which efficient solvers are available. The boundary conditions are then imposed by adding supplementary terms to the governing equations. Different penalization approaches are on the market, for example, surface and volume penalization techniques, immersed boundary methods using direct forcing and Lagrangian multipliers. For reviews on immersed boundary techniques, we refer to [13,10].

In the current work, we focus on the volume penalization method developed by Angot et al. [1] for imposing Dirichlet boundary conditions in viscous fluid flow. Physically, the boundary conditions correspond to no-slip conditions on the wall, *i.e.*, both the normal and the tangential velocity do vanish at the fixed wall. This penalization approach is physically motivated as walls or solid obstacles are modeled as porous media whose permeability tends to zero. Mathematically, it has also been justified. In [1,3] it was shown that the solution of the penalized Navier–Stokes equations converges towards the solution of the Navier–Stokes equations with no-slip boundary conditions, while the error depends on the penalization parameter. Various applications of the volume penalization method to impose Dirichlet boundary conditions can be found in the literature. Briefly summarizing, we can mention computations of confined hydrodynamic and magnetohydrodynamic turbulence, which can be found in [17] and [18,11], respectively. Fluid–structure interaction simulations have been carried out for moving obstacles [6] and for flexible beams [8]. Applications to the aerodynamics of insect flight in two and three space dimensions can be found in [7].

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Most of the developed penalization techniques deal with Dirichlet boundary conditions, and only few allow to impose Neumann conditions. Neumann boundary conditions in partial differential equations are encountered in many applications, for example when solving the Poisson equation for pressure in incompressible flows, to model adiabatic walls in heat transfer, or to impose no-flux conditions for passive or reactive scalars at walls. In [2] a review on the pure Neumann problem using finite elements is given and different techniques for solving the algebraic system are discussed. An extension of the volume penalization method [1] to impose Neumann or Robin boundary conditions has been presented in [14] and applied in the context of finite element or finite volumes [15]. In [5] we extended this method for pseudo-spectral discretizations and applied it to scalar mixing in incompressible flow for fixed and also for moving geometries imposing no-slip conditions for the velocity and no-flux conditions for the passive scalar field.

The fields of possible applications of the volume penalization method for imposing Neumann conditions in complex geometries are multifarious and large. For example, confined magnetohydrodynamic flow configurations can be studied imposing finite values of the current density at the wall, or convection problems which necessitate imposing a given heat flux at the boundary.

Motivated by the work of [9], where the properties of Fourier approximations of elliptic problems with discontinuous coefficients have been studied, we analyzed mathematically the penalized Laplace and Stokes operators with Dirichlet boundary conditions in [12] and verified the predicted convergence numerically. The aim of the present work is to generalize the approach developed in [12] and to analyze the penalized Laplace operator with Neumann boundary conditions. For a one-dimensional Poisson equation, we explicitly compute the penalization error by solving the penalized equation analytically. Discretizing the penalized equation using finite difference methods, we study the influences of both the numerical resolution and the value of the penalization parameter.

The outline of the paper is the following: First we consider the penalized Poisson equation in one space dimension with Neumann boundary conditions both analytically and numerically. Then, in Section 3 we study the eigenvalue problem of the penalized Laplace operator. Section 4 presents applications of the penalization method to solve the Poisson equation in two dimensions in a rectangular and a circular domain. Finally, some conclusions are drawn and some perspectives are given in Section 5.

2. Poisson equation with Neumann boundary conditions and penalization

2.1. Problem setting

We consider the one-dimensional Poisson equation

$$-w'' = f \quad \text{for } x \in (0, \pi) \tag{1}$$

completed with homogeneous Neumann boundary conditions, $w'(x=0) = w'(x=\pi) = 0$ and for $f(x) = m^2 \cos mx$, $m \in \mathbb{Z}$. The exact solution $w \in H^2(0,\pi)$ is given by $w(x) = \cos mx + C$, where $C \in R$ is an arbitrary constant, as the solution is not unique. Integrating Eq. (1) over $(0,\pi)$ yields the compatibility condition $\int_0^\pi f(x)\,dx = w'(x=\pi) - w'(x=0) = 0$ which has to be satisfied to guarantee the existence of a solution.

Following [5], the penalized Poisson equation reads

$$-d_x\big((1-\chi) + \eta\chi\big)\,d_x v = f \quad \text{for } x \in (0, 2\pi) \tag{2}$$

where $\eta > 0$ is the penalization parameter and χ the mask function defined by

$$\chi(x) = \begin{cases} 0 & \text{for } 0 < x < \pi \\ 1/2 & \text{for } x = 0 \text{ or } x = \pi \\ 1 & \text{elsewhere} \end{cases} \tag{3}$$

The domain $\Omega_f =]0, \pi[$, also called fluid domain, is imbedded into the larger domain $\Omega =]0, 2\pi[$ imposing now periodic boundary conditions at the boundary. Thus we have $\Omega = \Omega_f \cup \Omega_s$, where Ω_s is the penalization domain, also called solid domain.

2.2. Analytic solution of the one-dimensional penalized equation

The penalized Poisson equation (2) can be solved analytically in each sub-domain, i.e.,

$$-v'' = f \quad \text{for } x \in]0, \pi[\tag{4}$$

$$-\eta v'' = 0 \quad \text{for } x \in]\pi, 2\pi[\tag{5}$$

and accordingly we obtain

$$v(x) = \begin{cases} \cos mx + A_1 x + A_2 & \text{for } x \in]0, \pi[\\ B_1 x + B_2 & \text{for } x \in]\pi, 2\pi[\end{cases} \tag{6}$$