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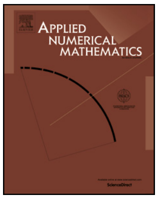

A relaxation Riemann solver for compressible two-phase flow with phase transition and surface tension

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ABSTRACT

The dynamics of two-phase flows depend crucially on interfacial effects like surface tension and phase transition. A numerical method for compressible inviscid flows is proposed that accounts in particular for these two effects. The approach relies on the solution of Riemann-like problems across the interface that separates the liquid and the vapour phase. Since the analytical solutions of the Riemann problems are only known in particular cases an approximative Riemann solver for arbitrary settings is constructed. The approximative solutions rely on the relaxation technique.

The local well-posedness of the approximative solver is proven. Finally we present numerical experiments for radially symmetric configurations that underline the reliability and efficiency of the numerical scheme.

1. Introduction

We consider the direct numerical simulation of a homogeneous compressible fluid that can appear in a liquid and in a vapour state. In particular we are interested in inviscid two-phase flows that account for surface tension as well as for mass exchange by evaporation and condensation.

The compressible hydrodynamics in the bulk phases is governed for the inviscid case by the Euler equations. The two-phase modelling is much more challenging because possible curvature and phase transition effects induce a complex transfer of momentum and energy through the interface. We follow here a sharp interface approach such that the spatial domain is partitioned into two bulk regions by a free boundary. The flow equations in each bulk region are coupled by appropriate trace conditions. The sharp interface approach is classical in multiphase fluid dynamics, and many numerical methods have been suggested. The dynamics in the bulk phases is given by the Euler equations and the local evolution of the phase boundary is determined from an interface model. Preliminary work in this direction can be found in [9,12], and for a comparable situation in porous media flow in [13].

Let us give an outline of this paper's content. In Section 2 we present the full mathematical model as a free boundary value problem in arbitrary spatial dimension. We consider the physically most relevant case of slow subsonic phase boundaries such that a Gibbs–Thomson like relation has to be added to the classical coupling conditions (conservation of mass, dynamical Young–Laplace law). Two choices for these conditions that determine the evolution of the interface are proposed.

As an interface model we obtain a planar Riemann problem, independent of the original spatial dimension. The initial states for the Riemann problem are in different phases. Such Riemann problems have been intensively studied in the last

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twenty years (see [14] for the general theory and [10] for a recent contribution). However, there are no explicit results for arbitrary pressure relations, kinetic relations and curvature dependent flow. Therefore we will follow [6] and construct in Section 3 an approximative Riemann solver which belongs to the class of relaxation solvers. This approximate Riemann solver is the major new contribution of this paper. With Theorems 3.1 and 3.2 we give basic well-posedness statements for the two kinetic relations under consideration. We believe however that the relaxation approach remains effective for even more general kinetic relations. Moreover the relaxation approximation can be applied to any kind of underlying pressure law including tabularized ones.

In Section 4 we discuss the complete numerical method for radially symmetric domains. The extension to arbitrary configurations in multiple space dimensions will be presented elsewhere. The overall method is summarised in Algorithm 4.2. Most notably the algorithm guarantees the conservation of mass. Finally in Section 5 we show numerical results. Convergence studies and long term simulations demonstrate the reliability of the overall method. In particular we show that the numerical method dissipates the associated physical entropy. Furthermore we present a detailed study on curvature effects, and we compare the mass transfer across the interface for different mobilities in the kinetic relations.

2. The mathematical model

2.1. A free boundary value problem for compressible liquid–vapour flow

As the basic modelling approach we consider a sharp interface ansatz. To introduce the precise setting let $\Omega \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ be an open bounded set. For any $t \in [0,T]$, $T > 0$, we assume that Ω is partitioned into the union of two open sets $\Omega_{\mathrm{vap}}(t)$, $\Omega_{\mathrm{liq}}(t)$, which contain the two bulk phases, and a hypersurface $\Gamma(t)$ – the sharp interface –, that separates the two spatial bulk sets. We restrict ourselves to isothermal motion at constant temperature $\theta > 0$, and let the fluid be inviscid. In the spatial–temporal bulk sets $\{(\boldsymbol{x},t) \in \Omega \times (0,T) \,|\, \boldsymbol{x} \in \Omega_{\mathrm{vap}}(t) \cup \Omega_{\mathrm{liq}}(t)\}$ the dynamics of the fluid is then governed by the hydromechanical system

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho\, \boldsymbol{v}) &= 0, \\ (\varrho\, \boldsymbol{v})_t + \operatorname{div}\big(\varrho\, \boldsymbol{v} \otimes \boldsymbol{v} + \tilde{p}(\varrho)\, \boldsymbol{I}\big) &= \boldsymbol{0}. \end{aligned} \tag{1}$$

Here $\varrho = \varrho(\boldsymbol{x},t) > 0$ denotes the unknown density field and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t) = (v_1(\boldsymbol{x},t), \cdots, v_d(\boldsymbol{x},t))^t \in \mathbb{R}^d$ the unknown velocity field. The pressure $\tilde{p} = \tilde{p}(\varrho)$ is a given scalar function and $\boldsymbol{I} \in \mathbb{R}^{d\times d}$ the d-dimensional unit matrix.

Before we proceed with the thermodynamical set-up for (1) let us add initial and boundary conditions. We fix the initial position of the interface $\Gamma(0)$ and assume for initial density ϱ_0 and velocity field $\boldsymbol{v}_0$

$$\varrho(\boldsymbol{x},0) = \varrho_0(\boldsymbol{x}), \qquad \boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{vap}}(0) \cup \Omega_{\mathrm{liq}}(0). \tag{2}$$

For the sake of simplicity we exclude flow across the boundary $\partial\Omega$, i.e.

$$\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{3}$$

where $\boldsymbol{n}$ is the outer normal to $\partial\Omega$.

We denote the specific volume by $\tau = 1/\varrho$ and remark that the pressure $p(\tau) = \tilde{p}(1/\tau)$ is related to the Helmholtz free energy $\psi = \psi(\tau)$ and the chemical potential $\mu = \mu(\tau)$ by

$$p(\tau) = -\psi'(\tau) \quad \text{and} \quad \mu(\tau) = \psi(\tau) + p(\tau)\tau. \tag{4}$$

As a prototype example we consider here the van der Waals pressure

$$\tilde{p}(\varrho) = \frac{R\,\theta\,\varrho}{1 - b\varrho} - a\varrho^2 \quad \text{for } \varrho \in \big(0, b^{-1}\big), \tag{5}$$

with positive constants $R, b, a > 0$. The constant subcritical temperature $\theta > 0$ is chosen such that $\tilde{p}$ is non-monotone. We refer to Fig. 1 for some illustration. Note that the pressure $\tilde{p}$ is increasing in the intervals $\tilde{\mathcal{A}}_{\mathrm{vap}} := (0, \varrho_{\mathrm{vap}}^{\mathrm{spinod}})$ and $\tilde{\mathcal{A}}_{\mathrm{liq}} := (\varrho_{\mathrm{liq}}^{\mathrm{spinod}}, b^{-1})$ which define the vapour and the liquid phase. For later use we introduce also $\mathcal{A}_{\mathrm{vap}} := (1/\varrho_{\mathrm{vap}}^{\mathrm{spinod}}, \infty)$

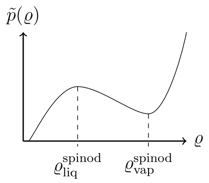

Fig. 1. Van der Waals pressure function $\tilde{p} = \tilde{p}(\varrho)$.