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Numerical solution of a multidimensional sedimentation problem using finite volume-element methods

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A R T I C L E I N F O

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A B S T R A C T

We are interested in the reliable simulation of the sedimentation of monodisperse suspensions under the influence of body forces. At the macroscopic level, the complex interaction between the immiscible fluid and the sedimentation of a compressible phase may be governed by the Navier–Stokes equations coupled to a nonlinear advection–diffusion–reaction equation for the local solids concentration. A versatile and effective finite volume element (FVE) scheme is proposed, whose formulation relies on a stabilized finite element (FE) method with continuous piecewise linear approximation for velocity, pressure and concentration. Some numerical simulations in two and three spatial dimensions illustrate the features of the present FVE method, suggesting their applicability in a wide range of problems.

1. Introduction

Gravitational sedimentation of small particles dispersed in a viscous fluid is a phenomenon widely observed in many engineering applications and natural systems. Such a process, where gravity drives the separation of the suspension into a clear supernatant liquid and a consolidated sediment, is typically employed in e.g. the solid–liquid separation of suspensions in mineral processing and wastewater treatment. Historically, many sedimentation models were based on the one-dimensional sedimentation theory by Kynch [26], which basically consists in describing the solids concentration as a function of vessel depth and time, governed by a scalar, nonlinear hyperbolic conservation law. This prototype model, along with a large number of tailored variants, have been proven to be accurate enough to represent some specific industrial scenarios such as clarifying–thickening processes and mixture of polydisperse suspensions (see e.g. [15,17]). Here we are more interested in multidimensional models that are able to predict some additional effects, such as the direct influence of bulk flow and boundary conditions. This necessarily entails the solution of the Navier–Stokes equations for the flow field of the mixture, increasing the complexity of the mathematical description of a strongly coupled system and also making difficult to solve these equations numerically. We focus our study in the particular case of batch sedimentation in closed channels exhibiting different geometries. The phenomenon of enhanced gravity settling in inclined channels, known as the Boycott effect, was first reported in [9] in the context of erythrocyte sedimentation. In such a context, the process of settling is enhanced when the vessel is tilted from the gravity force direction. A theoretical study of this phenomenon was later developed in

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the framework of fluid mechanics [24]. Since many years, the performance of inclined settlers has been described using the Ponder–Nakamura–Kuroda theory (see [36,35,21]), which does not consider the kinetics of the fluid motion. In [1] the authors generalized this theory by including a continuity equation for each phase, solid and fluid, an equation to describe the movement of the suspension as a continuum, and constitutive relationships between the velocities of each phase. In this context, our goal is to describe at least some components in the overall performance of inclined settlers such as the behavior of the layer of clear fluid along the underside of the inclined wall, which is accelerated upwards, and that enhances the sedimentation process. Even if multidimensional models of batch sedimentation show that the assumed one-dimensional nature of the concentration profile holds true in many scenarios, this is not the case for the velocity. Depending on the flow regime, it can drive the dynamics of the solids concentration and a number of variations in the flow field can be observed.

Besides experimental and theoretical investigations on the physics of sedimentation processes, numerical simulation is still an important tool to study the behavior of these phenomena. It provides an inexpensive way to try different configurations, nonstandard settings and special conditions for a given process. An important amount of work has been devoted to the determination of exact and numerical solution of related equations (see e.g. [8] and the references therein). Most of these contributions focus in the one-dimensional case, where all flow variables are assumed horizontally invariant and, subsequently, flow properties and boundary conditions can be replaced by modifications to the flux function, implying that only the concentration equation needs to be solved. More general models and solvers in two or three spatial dimensions include e.g. [31,32,27,13], where numerical studies of batch sedimentation in rectangular channels have been presented. Due to the nonlinearity and degeneracy of the concentration equation, traditional approaches are often unreliable and nonstandard techniques such as mixed or hybrid methods are needed. These difficulties also arise in e.g. the modelling of multiphase flows in porous media (see [4,29,40,37] and the references therein for the implementation and analysis of mixed finite elements/finite volumes of closely related equations).

Continuity equations can be solved numerically by a large variety of methods, each of them featuring some desirable properties. For instance, classical finite differences and finite volumes in structured meshes are straightforward to implement and can allow for conservative approximations, whereas finite element methods are robust and permit natural derivation of error estimates provided that the solution has enough regularity. Many modifications and combinations of these classical methods have been proposed to solve flow problems. Here we focus on one of these hybrid strategies, the finite volume element (FVE) method, that goes back to the early works [18,6]. Several variants of the FVE method are available from the literature, but irrespective of the specific form at hand, the main idea is that local conservativity is inherited from the finite volume part of the method, while maintaining the versatility and systematic error analysis in the L^2-norm. The FVE introduced in this paper is a special class of Petrov–Galerkin methods where the trial function spaces are connected with the test function spaces associated with the dual partition induced by the control volumes [38,28]. In summary, the method is able to effectively treat arbitrarily complex geometries and unstructured meshes, a variety of boundary conditions, and features local conservation and front capturing properties.

Works closely related to the present paper include the FVE method applied to Boussinesq equations analyzed in [30], the hybrid FE–FV scheme for incompressible flows presented in [19] and the ones for shear dependent viscoelastic flows from [34,46], the FV multiresolution method proposed in [13], and the stabilized FVE formulation for sedimentation problems in axisymmetric domains introduced in [14]. Our contribution represents an extension to the formers in that we consider flocculated suspensions (which translates in adding a degenerate diffusive term to the concentration equation), and to the latter in the sense that we cover the full three-dimensional case, and employ the Navier–Stokes equations for describing the flow. In addition, we use only one dual mesh, an interior penalty stabilization, and our formulation is based on piecewise continuous finite elements for all fields. The convergence analysis of (a regularized version of) the problem will be postponed for a future contribution.

The remainder of this paper is organized as follows. In Section 2 we recall some basic notation and state the model problem, specifying the conservation equations, constitutive relations, and weak formulation. The fully discrete FVE formulation is derived in Section 3, and Section 4 contains several numerical results illustrating the physical behavior of the system and the accuracy and robustness of the proposed FVE method. Concluding remarks and perspectives are collected in Section 5.

2. Preliminaries and statement of the initial–boundary value problem

We will consider the usual notation for Sobolev spaces $W^{m,k}(\Omega)$. The norm of $H^k(\Omega) = W^{k,2}(\Omega)$ is denoted by $\|\cdot\|_{k,\Omega}$ and the subscript Ω will be omitted unless otherwise specified.

The governing equations for the sedimentation–consolidation process in an immiscible fluid can be written as follows (see e.g. [11,15,22]):

$$\partial_t \phi + \operatorname{div} \boldsymbol{f}(\phi, \boldsymbol{u}) = \Delta A(\phi), \tag{2.1a}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mathbf{div}\big(\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{u}) - \lambda p \mathbf{I}\big) = \boldsymbol{g}(\phi), \tag{2.1b}$$

$$\lambda \operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega, \quad t > 0. \tag{2.1c}$$

The problem is defined for a given Lipschitz continuous domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with polyhedral boundary $\partial\Omega$ and outward pointing normal $\boldsymbol{n}$. The unknowns are the local solids concentration ϕ, the local volume-average velocity of the