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A hybrid discontinuous Galerkin method for advection–diffusion–reaction problems

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ABSTRACT

A hybrid discontinuous Galerkin (HDG) method for the Poisson problem introduced by Jeon and Park can be viewed as a hybridizable discontinuous Galerkin method using a Baumann–Oden type local solver. In this work, an upwind HDG method with super-penalty is proposed to solve advection–diffusion–reaction problems. A super-penalty formulation facilitates an optimal order convergence in the L^2 norm as well as the energy norm. Several numerical examples are presented to show the performance of the method.

1. Introduction

In this paper, a hybrid discontinuous Galerkin method introduced by Y. Jeon and E.-J. Park [17] is applied to advection–diffusion–reaction equations. To simplify our discussion, we assume that the domain Ω is a simply connected and bounded polygon with the boundary Γ. Let us consider the following advection–diffusion–reaction equations with the homogeneous Dirichlet boundary condition,

$$\begin{aligned} bu + \boldsymbol{a}\cdot\nabla u - \epsilon\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $b \geq 0$ and $\epsilon \geq 0$ are assumed to be constant. The advection vector field $\boldsymbol{a} : \Omega \to \mathbb{R}^2$ satisfies the *divergence free* condition and it belongs to the function space $W^1_\infty(\Omega)$. The function $f : \Omega \to \mathbb{R}$ is a suitably regular source term.

It is well-known that the standard finite element method yields numerical solutions with non-physical oscillation when advection dominated advection–diffusion problems are considered. Therefore, various stabilization techniques such as upwind, penalty, or exponentially fitted bases are often required to treat the problem.

In this paper, we suggest a discontinuous Galerkin (DG) approach using the upwind and penalty type stabilization. The DG method was introduced by Reed and Hill [21] in order to solve hyperbolic problems. Since then, there has been a number of researches on the DG methods for hyperbolic and nearly hyperbolic problems [3,7,11,14]. Also, the DG approach

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provides efficient numerical schemes for purely elliptic problems. For details of DG, local DG (LDG), interior penalty DG, hybridizable DG, and related methods, we refer to [2,6,8–10,12,25] and references therein.

Recently, the cell boundary element (CBE) method proposed by Jeon and his coworkers [15–19] produces flux conserving numerical solutions for elliptic problems. The CBE method can be viewed as a hybridized discontinuous Galerkin method with the Baumann–Oden type local solver [4]. We apply the CBE approach to the advection–diffusion–reaction problems with the upwind and penalty stabilization. In brief, our approach is composed of following steps:

Step 1. Introduce a trace $\lambda = u|_{K_h}$ on the *skeleton*.

Step 2. Set $u = u_\lambda + u_f$, where u_λ and u_f are solutions for the local advection–diffusion–reaction problems

$$\begin{aligned} &bu_\lambda + \boldsymbol{a}\cdot\nabla u_\lambda - \epsilon\Delta u_\lambda = 0 \quad \text{in } T, \qquad u_\lambda = \lambda \quad \text{on } \partial T, \\ &bu_f + \boldsymbol{a}\cdot\nabla u_f - \epsilon\Delta u_f = f \quad \text{in } T, \qquad u_f = 0 \quad \text{on } \partial T. \end{aligned} \tag{1.2}$$

The system is to be solved by an upwind method combined with penalty stabilization for each basis of λ.

Step 3. Use the flux continuity at cell interfaces to obtain a globally coupled system of equations in unknowns λ only.

There exist several methods in this direction: the hybridizable DG is based on the LDG local solver [9] and the interface stabilization method by Wells is based on the symmetric interior penalty local solver [24]. Interesting features of our formulation are in order. First, nonsymmetric interior penalty approach has intrinsic stability. Second, a super-penalty facilitates the optimal convergence in the L^2 norm as well as the energy norm. A super-penalty formulation has been studied in [5,13,22,23].

Note that the nonsymmetric interior penalty Galerkin (NIPG) method is proposed as a stabilized discontinuous Galerkin method of Oden, Babuska and Baumann [20] for diffusion problems. Since the NIPG method is not adjoint-consistent, it has some difficulty in error analysis in the L^2 norm [13,22,23]. Our analysis is related to the work of Wells [24] and covers both symmetric [24] and nonsymmetric [17] penalty formulation. Moreover, while convergence analysis in [24] is done only in the pure diffusion and the pure advection cases, our analysis covers general advection–diffusion problems.

The remainder of the paper is organized as follows. In the next section, triangulations, a skeleton of a mesh, related function spaces and norms are introduced. In Section 3, we introduce the penalized upwind CBE method. In Section 4, we investigate stability and convergence property of our formulation. Using a duality argument and a super-penalty formulation, we establish optimal error estimates for advection–diffusion–reaction equations. In the final section, numerical experiments confirming our theory are presented. Several numerical experiments for the advection–diffusion–reaction equations are performed to investigate the effect of the penalty parameters and small diffusion.

2. Preliminary

Let Ω be a simply connected and bounded polygonal domain with the boundary Γ. Let us denote the shape regular, quasi-uniform triangulation of Ω with $h = \max_{T\in\mathcal{T}_h} h_T$ by $\mathcal{T}_h$, where h_T is a diameter of an element T. The set of edges in $\mathcal{T}_h$ is denoted by E_h. The *skeleton* of a triangulation $\mathcal{T}_h$ is $K_h = \bigcup_{e\in E_h} e$, and the interior skeleton is denoted by $K_h^0 = K_h \setminus \Gamma$.

Let T be an element in $\mathcal{T}_h$. The boundary of T is denoted by ∂T, and $\boldsymbol{n}$ is the outward unit normal vector to the boundary ∂T. The outflow boundary of T is the part of ∂T on which $\boldsymbol{a}\cdot\boldsymbol{n} \geq 0$ and is denoted by ∂T_+. The inflow boundary of T is the part of ∂T on which $\boldsymbol{a}\cdot\boldsymbol{n} \leq 0$ and is denoted by ∂T_-.

For a set $\mathcal{O} \subset \mathbb{R}^2$ ($\mathcal{O}$ can be an element T, its boundary ∂T, the whole domain Ω, or its boundary Γ), $H^s(\mathcal{O})$ denotes the usual Hilbert space with $s \in \mathbb{R}$ and $H_0^s(\mathcal{O})$ denotes the subspace with zero trace on Γ. Let us denote the norms of $H^s(\mathcal{O})$ and seminorms of $H^s(\mathcal{O})$ by $\|\cdot\|_{s,\mathcal{O}}$ and $|\cdot|_{s,\mathcal{O}}$, respectively. The Hilbert space $H^s(\mathcal{T}_h) = \prod_{T\in\mathcal{T}_h} H^s(T)$ is equipped with the norm, $\|u\|_{s,h} := (\sum_{T\in\mathcal{T}_h} \|u\|_{s,T}^2)^{1/2}$ for $s \geq 0$. The discrete inner product is defined as $(\cdot,\cdot)_h = \sum_{T\in\mathcal{T}_h} (\cdot,\cdot)_T$. The pair $(\cdot,\cdot)$ represents the L^2 inner product on T or Ω, and $\langle\cdot,\cdot\rangle$ represents the L^2 inner product on K_h, ∂T or Γ.

Let us introduce function space on the skeleton K_h.

$$H_0^{1/2}(K_h) = \{u|_{K_h} : u \in H_0^1(\Omega)\}. \tag{2.1}$$

The function spaces $P_k(T)$ is the space of standard Lagrange polynomial functions of order $k \geq 1$ on T. Then

$$V_{k,h} := \bigoplus_{T\in\mathcal{T}_h} P_k(T), \quad P_k(\Omega) = \{p \in C(\Omega) : p \in V_{k,h}\}, \tag{2.2}$$

and

$$\Lambda_{k,h} := P_k(\Omega)|_{K_h}, \qquad \Lambda_k(\partial T) = P_k(T)|_{\partial T}. \tag{2.3}$$

The function space $\Lambda_{k,h}^0$ represents the subspace with zero trace on Γ.