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Reconstructions that combine interpolation with least squares fitting ☆

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ABSTRACT

We develop a reconstruction that combines interpolation and least squares fitting for point values in the context of multiresolution *a la Harten*. We study the smoothness properties of the reconstruction as well as its approximation order. We analyze how different adaptive techniques (*ENO*, *SR* and *WENO*) can be used within this reconstruction. We present some numerical examples where we compare the results obtained with the classical interpolation and the interpolation combined with least-squares approximation.

1. Introduction

Multiresolution representations have become effective tools for analyzing the information contents of a given signal. In this respect, the development of the theory of wavelets (Meyer [18], Cohen [6] and Daubechies [7]) has been a giant leap towards local scale decompositions with many applications in areas such as image compression, denoising, pattern recognition, photo recovery and in fields as diverse as Physics, Medicine or Engineering.

Multi-scale techniques do have an important role in Numerical Analysis. A wavelet type decomposition of a function is used to reduce the cost of many numerical algorithms by either applying it to the operator to obtain an approximate sparse form of it [5,10,15,2] or to the numerical solution itself to obtain an approximate reduced representation in order to solve for less quantities [13].

The building block of the wavelet theory is a square-integrable function whose dilates and translates form an orthonormal base of the space of square-integrable functions. Such uniformity leads to conceptual difficulties in extending wavelets to bounded domains and general geometries.

A combination of ideas from multigrid methods, numerical solution of conservation laws, hierarchical bases of finite element spaces, subdivision schemes of CAD (Computer-Aided Design) and, of course, the theory of wavelets led A. Harten to the development of a "General Framework" for multiresolution representation of discrete data [12,13].

Multiresolution representations *a la Harten* are constructed using two operators, decimation and prediction, which connect adjacent resolution levels. In turn, these operators are defined with two basic building blocks: the discretization and reconstruction operators. The former obtains discrete information from a given signal (belonging to a particular function

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space) and the latter produces an "approximation", in the same function space, from the discrete information contents of the original signal.

Because of the essential role played by the discretization and reconstruction operators in Harten's framework, building multiresolution schemes that are appropriate for a given application becomes a task which is very familiar to a numerical analyst. The first one identifies a sense of discretization (point values, cell averages, etc.) which is appropriate for the given application. The second one solves a problem in approximation theory in this context.

The discretization operator specifies the process of generation of discrete data and, thus, it determines the nature of the discrete data to be analyzed. When reinterpreted within Harten's framework, the discretization operator in the wavelet theory is obtained by taking local averages against the scaling function.

The strict requirements of the wavelet theory rule out many scaling functions that provide, nevertheless, appropriate discretization settings in many situations. For example weighted averages against the δ-function lead to point value discretizations, a natural discretization procedure for continuous functions which is widely used within the Numerical Analysis community. It is shown in [4,3] how to obtain multi-scale decompositions in the point value context.

In this paper we develop a reconstruction that combines interpolation and least squares approximation for point values in the context of multiresolution *a la Harten*. We study the smoothness properties of the reconstruction as well as its approximation order.

The reconstruction, hence the prediction, operator needs not to be linear. This allows us to introduce adaptivity, like *ENO* (*Essentially Non-Oscillatory*, [14]), *SR* (*Subcell Resolution*, [12]) and *WENO* (*Weighted ENO*, [17]), in the reconstruction presented here.

The paper is organized as follows: we recall in Section 2 the discrete framework for multiresolution introduced by Harten, focusing on point values discretizations. We introduce the *interpolation–approximation* reconstruction in Section 3, where some of their properties are analyzed. In this section we also present how this reconstruction can be combined with non-linear techniques. In Section 4 we show some results that compare the performance of the reconstruction introduced in this paper with the classical interpolatory reconstruction and in Section 5 we present the conclusions.

2. Harten's multiresolution in the point value framework

Here we summarize the basics of Harten's multiresolution framework for the point value approach. A more extensive description can be found in [4,13].

For each resolution level (from finest to coarsest) $k = L, \dots, 0$, let us consider the following partition of the unit interval $[0,1]$: $X^k = \{x_i^k\}_{i=0}^{N_k}$, $x_0^k = 0$, $x_i^k = i \cdot h_k$, $h_k = \frac{1}{N_k}$, $N_k = 2^k \cdot N_0$, where N_0 is some fixed natural. Note that X^{k+1} is obtained from X^k by adding the middle points of the subinterval $[x_{i-1}^k, x_i^k]$. That is, $x_{2i-1}^{k+1} = (x_{i-1}^k + x_i^k)/2$.

Let us assume that the data, $\{f_i^k\}_{i=0}^{N_k}$, represents the point values of a given piecewise smooth real function f at the points of the different resolution levels, that is, $f_i^k = f(x_i^k)$.

Notice that $x_i^k = x_{2i}^{k+1}$, which indicates the procedure for obtaining the discrete information of a function at a certain resolution level from the information from the higher resolution level:

$$f_i^k = f_{2i}^{k+1}, \quad i = 0, \dots, N_k. \tag{1}$$

If we want to know the information corresponding to a higher resolution level, we will use the above expression for the values corresponding to even indexes, and *predict* by interpolation an approximation to the values corresponding to odd indexes. If $q_{i,n_l,n_r}^r(x; f^k)$ denotes the polynomial that interpolates the data at some stencil around x, i.e. the polynomial of degree $r = n_r + n_l - 1$ such that $q_{i,n_l,n_r}^r(x_j^k; f^k) = f_j^k$, $j = i - n_l, \dots, i + n_r - 1$ $(n_l, n_r \geq 1)$, then we define

$$\overline{\mathcal{IP}}_{n_l,n_r}^r(x; f^k) = q_{i,n_l,n_r}^r(x; f^k), \quad x \in [x_{i-1}^k, x_i^k], \; i = 1, \dots, N_k,$$

and

$$(P_k^{k+1} f^k)_{2i-1} = \overline{\mathcal{IP}}_{n_l,n_r}^r(x_{2i-1}^{k+1}; f^k) = \overline{\mathcal{IP}}_{n_l,n_r}^r(x_{i-\frac{1}{2}}^k; f^k). \tag{2}$$

The prediction error is defined as $e_i^{k+1} = f_i^{k+1} - \overline{\mathcal{IP}}_{n_l,n_r}^r(x_i^{k+1}; f^k)$, $i = 0, \dots, N_{k+1}$, where $e_i^{k+1} = 0$, $\forall i$ such that $x_i^{k+1} \in X^k$. The coefficients $d^{k+1} = \{d_i^{k+1}\}$ are $d_i^{k+1} = e_{2i-1}^{k+1}$ for $i = 1, \dots, N_k$.

With these expressions, we will be able to build a multiscale representation of a given set of discrete data, $f^L = \{f_i^L\}_{i=0}^{N_L}$, by a direct transformation. Reciprocally, we will recover the original discrete data from its multiscale representation by an inverse transformation. The algorithms for direct (3) and inverse (4) transformation are:

$f^L \to Mf^L = \{f^0, d^1, \dots, d^L\}$ (Direct)

$$\left\{\begin{array}{lll} \textbf{for } k & = L-1, \dots, 0 & \\ \quad f_i^k & = f_{2i}^{k+1}, & 0 \leq i \leq N_k \\ \quad d_i^{k+1} & = f_{2i-1}^{k+1} - (P_k^{k+1} f^k)_{2i-1}, & 1 \leq i \leq N_k \\ \textbf{end} & & \end{array}\right. \tag{3}$$