Applied Numerical Mathematics 97 (2015) 69–87

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Existence and uniqueness of optimal solutions for multirate partial differential algebraic equations

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ARTICLE INFO

Article history:
Received 18 December 2014
Received in revised form 11 May 2015
Accepted 31 July 2015
Available online 5 August 2015

Keywords:
Multirate partial differential algebraic equation
Optimisation
Optimal control
Method of characteristics
Radio frequency applications

ABSTRACT

The numerical simulation of electric circuits including multirate signals can be done by a model based on partial differential algebraic equations. In the case of frequency modulated signals, a local frequency function appears as a degree of freedom in the model. Thus the determination of a solution with a minimum amount of variation is feasible, which allows for resolving on relatively coarse grids. We prove the existence and uniqueness of the optimal solutions in the case of initial-boundary value problems as well as biperiodic boundary value problems. The minimisation problems are also investigated and interpreted in the context of optimal control. Furthermore, we construct a method of characteristics for the computation of optimal solutions in biperiodic problems. Numerical simulations of test examples are presented.

1. Introduction

The numerical simulation of electric circuits applies models consisting of differential algebraic equations, see [7,11]. In radio frequency applications, the solutions exhibit multirate signals with amplitude modulation or frequency modulation at widely separated time scales. Thus a transient simulation of the dynamical systems becomes tedious and costly.

Alternatively, a multidimensional signal representation enables to decouple the time scales. Brachtendorf et al. [3] introduced a suitable model composed of multirate partial differential algebraic equations (MPDAEs). Either initial-boundary value problems (IBVPs) or biperiodic boundary value problems (BBVPs) have to be solved. The MPDAE model was investigated for the case of amplitude modulation in [1,4,14,21,23].

The approach was extended to include also the case of frequency modulation by Narayan and Roychowdhury [13]. Now a local frequency function appears as degrees of freedom in the MPDAE system. Thus additional constraints are required to identify an appropriate solution. On the one hand, the MPDAEs together with phase conditions or similar constraints were examined in [15–17,20,25]. On the other hand, the idea to determine the degrees of freedom by a minimisation goes back to Houben [8,9], where the aim is to guarantee an efficient multidimensional representation. A variational calculus yields necessary conditions for an optimal solution, which were derived in [18] for IBVPs and in [19] for BBVPs. A minimisation approach was also applied in [2,6] for IBVPs. A survey on all the cases above can be found in [22].

Although the MPDAE model has already been simulated successfully by strategies based on minimisation, the existence and uniqueness of solutions was not shown yet. In this paper, we prove the existence and uniqueness of minima with respect to functionals both for IBVPs and BBVPs. The proofs also produce necessary conditions for an optimal solution,

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http://dx.doi.org/10.1016/j.apnum.2015.07.004
0168-9274/

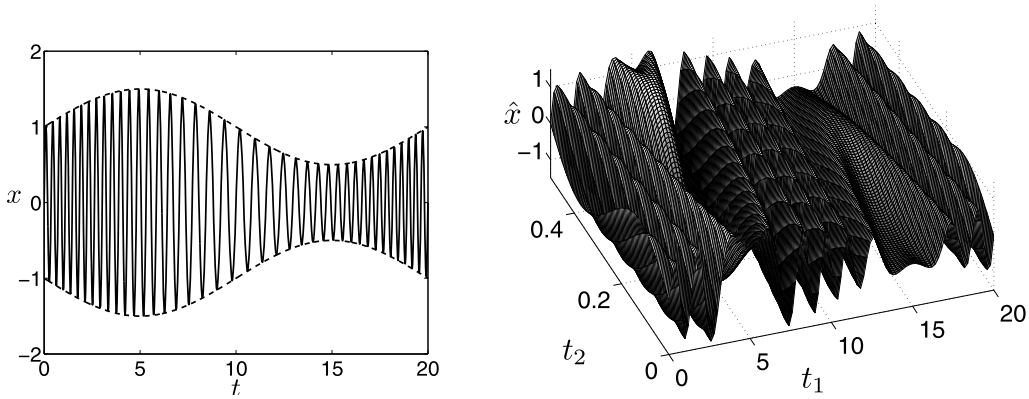

Fig. 1. Multitone signal x (left) and MVF $\hat{x}$ (right).

where the criteria from [18,19] are recovered and a new condition appears. In addition, the minimisation problems are interpreted in the context of optimal control. A method of characteristics yields the numerical solutions of BBVPs of the MPDAEs efficiently in the case of continuous phase conditions, see [15]. We modify the method of characteristics such that the necessary conditions for an optimal solution are included.

The paper is organised as follows. We outline the multidimensional signal model and the MPDAE system in Section 2. We define the minimisation problems and provide the results on existence and uniqueness in Section 3. Connections to optimal control are discussed. Furthermore, the method of characteristics is sketched. In Section 4, numerical simulations are presented for two test examples.

2. Multirate modelling

We outline a multidimensional signal model and the resulting partial differential algebraic equations.

2.1. Multivariate signal model

We consider high-frequency signals, which are both amplitude modulated and frequency modulated. In an example from [19], the multitone signal

$$x(t) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t\right)\right] \cdot \sin\left(\tfrac{2\pi}{T_2} t + \beta \sin\left(\tfrac{2\pi}{T_1} t\right)\right) \tag{1}$$

with $T_1 \gg T_2$ exhibits an amplitude modulation introduced by the parameter $0 < \alpha < 1$, whereas the parameter $\beta > 0$ defines the amount of frequency modulation. Fig. 1 (left) depicts the behaviour of the signal (1). We obtain a multivariate signal model, which is called the multivariate function (MVF), by the introduction of an own variable for each time rate. The formula (1) implies directly

$$\hat{x}(t_1, t_2) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \cdot \sin\left(\tfrac{2\pi}{T_2} t_2 + \beta \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right). \tag{2}$$

This MVF is biperiodic and thus, already determined by its values in the rectangle $[0, T_1] \times [0, T_2]$. The original signal (1) can be reconstructed via $x(t) = \hat{x}(t, t)$. Unfortunately, this MVF includes many oscillations in the domain of definition as shown in Fig. 1 (right). The number of oscillations even increases with the value of the parameter β. Hence the naive representation (2) is inappropriate in the case of frequency modulated signals.

To obtain an efficient representation, Narayan and Roychowdhury [13] propose to model the frequency modulation separately. Thus the MVF just includes the amplitude modulation part, which yields the biperiodic description

$$\hat{y}(t_1, t_2) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \cdot \sin(2\pi t_2), \tag{3}$$

where the second period is standardised to 1. Fig. 2 (left) illustrates the MVF (3) and we recognise a simple form in the domain of definition $[0, T_1] \times [0, 1]$. Thus an efficient representation is achieved. The frequency modulation part is specified by an additional time-dependent function

$$U(t) := \frac{t}{T_2} + \frac{\beta}{2\pi} \sin\left(\tfrac{2\pi}{T_1} t\right). \tag{4}$$

We perform the reconstruction of the original signal (1) using

$$x(t) = \hat{y}(t, U(t)). \tag{5}$$