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Carleman estimates for the regularization of ill-posed Cauchy problems

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ABSTRACT

This work is a survey of results for ill-posed Cauchy problems for PDEs of the author with co-authors starting from 1991. A universal method of the regularization of these problems is presented here. Even though the idea of this method was previously discussed for specific problems, a universal approach of this paper was not discussed, at least in detail. This approach consists in constructing of such Tikhonov functionals which are generated by unbounded linear operators of those PDEs. The approach is quite general one, since it is applicable to all PDE operators for which Carleman estimates are valid. Three main types of operators of the second order are among them: elliptic, parabolic and hyperbolic ones. The key idea is that convergence rates of minimizers are established using Carleman estimates. Generalizations to nonlinear inverse problems, such as problems of reconstructions of obstacles and coefficient inverse problems are also feasible.

1. Introduction

This is a survey of works dedicated to a universal method of the regularization of ill-posed Cauchy problems for PDEs of the second order. The idea of this method was originated in 1991 in works of Klibanov and Santosa [45] and of Klibanov and Malinsky [46]. Then this idea was explored in a number of publications of the author with coauthors as well as in works of Bourgeois with coauthors. Bourgeois has started to work on this topic in 2005 [15]. Corresponding references with comments are cited below. Until the current paper this idea was used only for specific ill-posed Cauchy problems. However, a *systematic study* of the universal approach for constructing regularization methods for ill-posed Cauchy problems for general PDEs of the second order, which is based on this idea, was not presented before. The latter is done here.

Our universal regularization method works for those PDEs, for which Carleman estimates are valid. On the first step of this approach such a Tikhonov functional is constructed, which is generated by the unbounded operator of the corresponding PDE. On the second step the convergence rate of minimizers of that functional is established using the Carleman estimate for the corresponding PDE operator. We now refer to the papers of the author with coauthors [27,29,47,51–53], which have explored this method.

Let $\Psi \subset \mathbb{R}^n$ be a bounded domain and A be a linear Partial Differential Operator (PDO) of the second order acting in Ψ. Likewise, assume that this operator admits a Carleman estimate. In fact, the class of such operators is quite broad. Indeed, currently Carleman estimates are derived for three main classes of PDOs of the second order: elliptic, parabolic and hyperbolic ones, see, e.g. books of Beilina and Klibanov [9], Isakov [36], Klibanov and Timonov [50], and Lavrentiev, Romanov and Shishatskii [62] as well as the paper of Triggiani and Yao [69]. Therefore, results of this paper are very general ones.

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Consider the Partial Differential Equation (PDE) $Au = f$ and an ill-posed Cauchy problem for it. Suppose that the Tikhonov functional for the solution of this problem is generated by the operator A. The current paper provides the positive answer for the following question: *Can the solution of this Cauchy problem be approximated via the minimization of this functional?*

Typically the regularization term is presented in the Tikhonov functional in a norm, which is stronger than the norm of the original space. Hence, we consider in our setting the domain of A as $D(A) = H^2(\Psi) \subset L_2(\Psi)$ and $A : H^2(\Psi) \to L_2(\Psi)$. Thus, in this specific context $H^2(\Psi)$ is a linear set in $L_2(\Psi)$ and $\overline{H}^2(\Psi) = L_2(\Psi)$, where the closure is taken in the norm of $L_2(\Psi)$. Thus, we consider A as an unbounded operator. As to the regularization theory for linear ill-posed problems with bounded linear operators, we refer to, e.g. the book of Ivanov, Vasin and Tanana [37].

First, we present below our universal approach in which the operator A of the original PDE generates the Tikhonov functional. Next, we specify our method for four (4) main classes of ill-posed Cauchy problems: Cauchy problems for elliptic PDEs, problems for hyperbolic and parabolic PDEs with the lateral Cauchy data and the initial boundary value problem for the parabolic PDE with the reversed time. In addition, we briefly outline in subsections 8.2, 8.3 extensions of this method to two important *nonlinear* inverse problems: inverse obstacle problems and coefficient inverse problems.

Concerning the applications of Carleman estimates to inverse problems, we refer to the method, which was first proposed by Bukhgeim and Klibanov [24,25,42] (papers [25,42] contain first full proofs). The method of [24,25,42] was originally designed for proofs of uniqueness theorems for Coefficient Inverse Problems (CIPs) with single measurement data, see, e.g. some follow up works of Bukhgeim [26], Klibanov [43,44], Klibanov and Timonov [50], surveys of Klibanov [54] and Yamamoto [71], as well as Sections 1.10 and 1.11 of the book of Beilina and Klibanov [9]. Later, this idea was extended to globally convergent numerical methods for CIPs, see works of the author with coauthors [13,48–50,55], the paper of Baudouin, de Buhan and Ervedoza [7] and subsection 8.3.

As it was stated above, in our universal regularization method Carleman estimates provide convergence rates of minimizers of those Tikhonov functionals. The intrinsic reason why this can be done is that Carleman estimates ensure Hölder stability estimates in certain subdomains for those Cauchy problems in the case of elliptic and parabolic PDEs, see, e.g. [36,50,54,62]. In the hyperbolic case the Carleman estimate provides a much stronger Lipschitz stability estimate in the whole domain, see this section below and Section 6. It turns out that the Hölder stability estimate in a subdomain is a certain analog of the estimate of the modulus of the continuity of the inverse operator. On the other hand, it is one of classical results of the theory of ill-posed problems that an estimate of the modulus of the continuity on a compact set of the inverse operator provides the rate of convergence of minimizers of the Tikhonov functional, see, e.g. books of Beilina and Klibanov [9], Engl, Hanke and Neubauer [33], Kabanikhin [38], Lavrentiev, Romanov and Shishatskii [62] and Tikhonov, Goncharsky, Stepanov and Yagola [68].

The first Tikhonov functionals for ill-posed Cauchy problems for PDEs, which were generated by operators of those PDEs, were constructed in the *pioneering work* of Lattes and Lions [61]. Lattes and Lions have called their approach the "Quasi-Reversibility Method" (QRM). Their book contains examples of many ill-posed Cauchy problems. They have presented two versions of the QRM. In the first version, additional terms with regularization parameters in them were introduced in those PDEs. In the second version, which is close to ours, strong formulations of those Cauchy problems were considered first. In such a formulation the fourth order operator A^*A is involved. Next, for elliptic and parabolic PDEs, weak variational formulations of equations with A^*A were considered. In the elliptic case, that variational formulation was equivalent with the minimization of a Tikhonov functional generated by the operator A. In the parabolic case, the original strong formulation led to an unnecessary term $(u(T), v(T))$ in the variational form, see the formula (3.6) on page 324 of [61]. Convergence theorems were proved in [61]. However, convergence rates were not established in [61] and Carleman estimates were not used. In addition, certain cut-off functions were used in [61], which is unnecessary.

First applications of the tool of Carleman estimates to the topic of ill-posed Cauchy problems for PDEs were done in papers of Klibanov and Santosa [45] and Klibanov and Malinsky [46]. As a result, first convergence rates of minimizers of those Tikhonov functionals were established in these references. Both these works have considered the variational form of the Tikhonov functional. In [45] the Cauchy problem for the Laplace operator was considered, see the paper of Cao, Klibanov and Pereverzev [27] for a continuation of [45]. Since in the elliptic case the Hölder stability estimate can be proved by the Carleman estimate only in a subdomain, then Hölder-like convergence rates of minimizers in [27,45] were established only in subdomains; also see Sections 2–5 below.

The paper [46] is the first one where the Lipschitz stability estimate in the entire time cylinder was proved for the hyperbolic equation with the lateral Cauchy data, using a Carleman estimate (see Theorem 6.1 in Section 6). Given the Carleman estimate, the Lipschitz stability estimate became possible basically because the hyperbolic equation can be stably solved in both directions of time: positive and negative. The Lipschitz stability estimate, in turn allows to establish Lipschitz-like convergence rates of minimizers of the corresponding Tikhonov functional in the entire time cylinder, see [46] and Section 6. There were several follow up works, which explored some modifications of the idea of [46] to prove the Lipschitz stability for the hyperbolic case. More precisely, those were works of Kazemi and Klibanov [41], Klibanov and Timonov [50], Isakov [36], Clason and Klibanov [29], Klibanov, Kuzhuget, Kabanikhin and Nechaev [52] and the survey [54]. While all these publications are about the case of the Euclidean geometry, the more general case of the Riemannian geometry was considered by Triggiani and Yao [69]. Lasiecka, Triggiani and Zhang [59,60] have extended this technique to the case of the Schrödinger equation.

It is shown in Section 7 that the original technique of [46] allows one to obtain the Lipschitz stability estimate, to construct the Tikhonov functional and to obtain the Lipschitz-like convergence rate of its minimizers for the problem of