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Exact Riemann solvers for conservation laws with phase change

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ABSTRACT

In this paper we consider the solid–solid phase transformation in martensitic materials and present two numerical procedures for solving exactly the Riemann problems of a 3×3 system of conservation laws [21]. A particular attention is given to the configurations of the phase boundaries. For a Riemann problem whose initial states are specified in different phases, we first assume that the phase boundary is stationary and then find the solution through an iteration method [24]. Configuration of the transition front is then determined based on this stationary-phase-boundary solution [12]. The solution with dynamic phase change is calculated by listing all the relations in the Riemann problem and solving the resulting nonlinear system. Another approach, which avoids solving this system, is also proposed where the solution is obtained by computing the intersection of two projection curves. A front capturing/tracking method [25] implementing these Riemann solvers is presented to approximate initial value problems with propagating transition fronts. This approach captures the phase boundary sharply without artificial smearing in the physically unstable region.

1. Introduction

Systems of conservation laws can be used to model phase transition problems. Examples include compressible multiphase flows as well as crystalline solids, such as shape-memory alloys, that admit more than one phases. In a phase transition problem, stress is not a monotonically increasing function of strain. The system becomes elliptic in the region where the stress-strain function decreases. We assume that this region is physically unstable hence the system is essentially hyperbolic.

In this work we consider a martensitic phase transition problem for longitudinal deformation of homogeneous bar with unit cross section [21]. The low-strain state and high-strain state of the material are also called α-phase and β-phase, respectively. The latent heat is nonzero and the phase boundary propagates slower than a shock or a rarefaction wave. Let $u(x,t)$ represent displacement of a reference point x at t, $w = u_x$ is strain and $v = u_t$ is the particle velocity. The standard balance of mass, linear momentum and energy for adiabatic motions take the form

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$$\begin{cases} w_t - v_x = 0, \\ v_t - \sigma_x = 0, \\ E_t - (\sigma v)_x = 0, \end{cases} \tag{1}$$

where $t \geq 0, -\infty < x < \infty$. σ and E are stress and total energy, respectively. $E = e + v^2/2$, where $e(w,s)$ is the specific internal energy (s is the specific entropy) and

$$\sigma = e_w(w,s).$$

When $\sigma_w(w,s) > 0$, the system (1) is hyperbolic with three characteristic speeds

$$\lambda = \left\{-\sqrt{\sigma_w}, 0, \sqrt{\sigma_w}\right\}.$$

When no phase transition occurs, the solution of the Riemann problem contains a backward wave, a contact discontinuity and a forward wave corresponding to the three characteristic families. The stress-strain relation is given by [21]

$$\sigma(w,T) = A + BT + K(w - w_a)(w - w_b)(w - \frac{1}{2}(w_b + w_a)), \tag{2}$$

where constants A, B, K, w_a and w_b satisfy that $A > 0$, $B > 0$, $K > 0$ and $w_b > w_a \geq 0$. We assume that the specific heat c is constant and the Helmholtz free energy is written as

$$f(w,T) = Aw + BTw + K\psi(w) - cT\ln\frac{T}{T_0}, \tag{3}$$

where

$$\psi(w) = \frac{w^4}{4} - \frac{1}{2}(w_a + w_b)w^3 + \frac{1}{2}w^2\left(\frac{w_a^2}{2} + 2w_a w_b + \frac{w_b^2}{2}\right) - w_a w_b\left(\frac{w_a + w_b}{2}\right)w,$$

such that

$$\psi'(w) = (w - w_a)(w - w_b)(w - \frac{1}{2}(w_b + w_a)).$$

The entropy is

$$s(w,T) = -Bw + c\ln\frac{T}{T_0} + c.$$

Solve for T to obtain that

$$T = T_0\exp\left(\frac{s + Bw - c}{c}\right),$$

and plug it into (2) to see that

$$\sigma = A + BT_0\exp\left(\frac{s + Bw - c}{c}\right) + K(w - w_a)(w - w_b)(w - \frac{1}{2}(w_b + w_a)). \tag{4}$$

By setting $\sigma = \sigma_0$, we have the entropy-strain relation at constant stress as

$$s = -Bw + c\left\{1 + \ln\frac{1}{BT_0}\left[\sigma_0 - A - K(w - w_a)(w - w_b)(w - \frac{1}{2}(w_b + w_a))\right]\right\}. \tag{5}$$

The internal energy is given by

$$\begin{aligned} e &= f + Ts \\ &= Aw + K\psi(w) + cT_0\exp\left(\frac{s + Bw - c}{c}\right). \end{aligned} \tag{6}$$

It is easy to verify the thermodynamic relation that

$$de = \sigma dw + Tds, \tag{7}$$

i.e., $\sigma = e_w$ and $T = e_s$.

In order to pick out the physically relevant solution, we need an entropy condition which imposes that the entropy increases across jump discontinuities. Let

$$F = D(s_+ - s_-), \tag{8}$$