



On the uniqueness and reconstruction for an inverse problem of the fractional diffusion process



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ABSTRACT

Consider an inverse problem for the time-fractional diffusion equation in one dimensional spatial space. The aim is to determine the initial status and heat flux on the boundary simultaneously from heat measurement data given on the other boundary. Using the Laplace transform and the unique extension technique, the uniqueness for this inverse problem is proven. Then we construct a regularizing scheme for the reconstruction of boundary flux for known initial status. The convergence rate of the regularizing solution is established under some *a priori* information about the exact solution. Moreover, the initial distribution can also be recovered approximately from our regularizing scheme. Finally we present some numerical examples, which show the validity of the proposed reconstruction scheme.

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1. Introduction

The inverse problems for the heat conduction process are of great importance in the applied area, and the aim is to determine unknown ingredients of the conduction system from some measurement data. Generally, the unknowns in the diffusion system are classified into three categories: media parameters, heat status in some part of the boundary, and the initial status of the diffusion process. The corresponding inverse problems are called as parameter identification problems, side-way problems and backward problems, respectively.

Of course, the properties of inverse problems depend on the system describing the diffusion process. In the case of classical heat conduction process, which means the diffusion process with respect to the time is described by the first order derivative of time t , the inverse problems have been researched thoroughly both in one dimensional and higher dimensional cases, we refer to [2–10,13,15,16] and the references therein.

On the other hand, the fractional order derivatives have found wide applications in describing some physical and industry processes, and consequently the direct problems for PDEs with the fractional-order partial derivatives have been researched extensively for the theoretical analysis and numerical algorithms. As a special application of fractional derivative describing the classical diffusion process, it was shown that the classical diffusion equation including second spatial derivatives and first time derivatives can be reduced to a simpler equation containing only a first-order spatial derivative and a-half order time derivative under some assumptions, e.g., [21,22]. The definition of fractional-order derivative in the Caputo sense requires that the direct problems should be solved in some different way, since the fractional order derivative is nonlocal. For recent researches, see [1,11,14,17,18,23,26,29].

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To our knowledge, the researches on the inverse problems related to the PDEs of fractional-order derivatives are still in the initial stage, and very little works are found. The difficulty comes from the fractional-order derivatives, which is essentially an integral with the kernel of weak singularity. For this new kind of derivative, some classical computation rules such as integration by parts become invalid. Henceforth, some standard methods for treating the inverse problems such as the Carleman estimates cannot be used any more. Such a difference makes the inverse problems for PDEs with fractional derivatives much difficult.

In this paper, we consider an inverse problem for the fractional diffusion process governed by fractional order derivative with respect to time t in one dimensional space. Our inverse problem is the side-way and the backward problem: we try to identify the heat flux on one side of the boundary and the initial status of the diffusion process simultaneously, from the measurement data specified on the other boundary.

More precisely, for given $\alpha \in (0, 1)$, we consider the following problem

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx}, & x \in (0, \pi), t > 0 \\ u_x(0, t) = a(t), u_x(\pi, t) = b(t), & t > 0 \\ u(x, 0) = \varphi(x), & x \in (0, \pi), \end{cases} \quad (1.1)$$

where we use the Caputo derivative defined as

$$\frac{d^\alpha}{dt^\alpha} g(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t g'(s) \frac{ds}{(t-s)^\alpha} \quad (1.2)$$

with $\Gamma(\cdot)$ the standard Γ -function.

If $a(t)$, $b(t)$ and $\varphi(x)$ are given, then this is a forward fractional diffusion problem. Now we consider the inverse problem for given $a(t)$ and the additional condition

$$u(0, t) = f(t), \quad t > 0. \quad (1.3)$$

We are concerned with the following problems:

- Is it possible to determine $b(t)$ and $\varphi(x)$ simultaneously from one pair given data $(a(t), f(t))$ in $t \in (0, \infty)$?
- How to construct an efficient recovering scheme from $(a(t), f(t))$ given approximately only in a finite time interval $[0, T]$? What is the numerical performance?

We should mention that in two dimensional case with classical heat diffusion process (i.e., $\alpha = 1$), the related problems have been considered by the Carleman estimate technique, see [25,28]. Also in the case of $\alpha = 1$, the uniqueness of recovering $u_x(x, \pi)$ and initial value has been proven in [9], while the numerical reconstruction of side value without the initial value is given in [25].

2. Uniqueness result

Firstly, we make a transform to express the original problem in an equivalent form. Let

$$u(x, t) = v(x, t) + \frac{1}{2\pi} [b(t) - a(t)]x^2 + a(t)x. \quad (2.1)$$

Then the new function $v(x, t)$ solves

$$\begin{cases} \frac{\partial^\alpha v}{\partial t^\alpha} = v_{xx} + F[a, b](x, t), & x \in (0, \pi), t > 0 \\ v_x(0, t) = 0, v_x(\pi, t) = 0, & t > 0 \\ v(x, 0) = \Phi(x), & x \in (0, \pi), \end{cases} \quad (2.2)$$

with

$$F[a, b](x, t) := \left(\frac{1}{2\pi} x^2 - x \right) \frac{\partial^\alpha a(t)}{\partial t^\alpha} - \frac{1}{\pi} a(t) + \frac{1}{\pi} b(t) - \frac{1}{2\pi} x^2 \frac{\partial^\alpha b(t)}{\partial t^\alpha}, \quad (2.3)$$

$$\Phi(x) := \varphi(x) + \frac{1}{2\pi} [a(0) - b(0)]x^2 - a(0)x. \quad (2.4)$$

The additional condition at $x = 0$ keeps unchanged:

$$v(0, t) = f(t), \quad t > 0. \quad (2.5)$$

The first problem that we are interested is: can we identify the side heat flux $b(t)$ and initial statue $\varphi(x)$ simultaneously from the system (2.2)–(2.5) for given $a(t)$ and $f(t)$?

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