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Method of infinite systems of equations for solving an elliptic problem in a semistrip

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ABSTRACT

Many problems of mechanics and physics are posed in unbounded (or infinite) domains. For solving these problems one typically limits them to bounded domains and finds ways to set appropriate conditions on artificial boundaries or use quasi-uniform grid that maps unbounded domains to bounded ones. Differently from the above methods we approach to problems in unbounded domains by infinite systems of equations. In this paper we develop this approach for an elliptic problem in an infinite semistrip. Using the idea of Polozhii in the method of summary representations we transform the infinite system of three-point vector equations to infinite systems of three-point scalar equations and obtain the approximate solution with a given accuracy. Numerical experiments for several examples show the effectiveness of the proposed method.

1. Introduction

Many problems of mechanics and physics are posed in unbounded (or infinite) domains. For solving these problems one usually restricts oneself to treat the problem in a bounded domain and tries to use available efficient methods for finding exact or approximate solutions in the restricted domain. But there arise questions: which size of restricted domain is enough and how to set conditions on the artificial boundary for obtaining approximate solutions with a good accuracy?

The simplest way to do this is to transfer the boundary conditions on infinity to the artificial boundary. This raw way may lead to large deviation of approximate solutions from the exact solution of the original problem. Therefore, instead of transferring the boundary condition on infinity without changes one tries to set appropriate conditions on the artificial boundary. These boundary conditions are called artificial or absorbing boundary conditions (ABC) since some "energy" is absorbed at the boundary [1]. In the problems for the wave equation they are often referred to as non-reflecting boundary conditions (NRBC). They are constructed with the objective to approximate the exact solution of the unbounded problem restricted to the bounded domain. The researches of ABCs attract attention from many specialists in mathematics, mechanics and physics. Before the 1980s some lower order NRBCs, e.g., the Engquist–Majda NRBCs [10] and the Bayliss–Turkel NRBCs [2] were proposed. Later, in the mid-1990s the perfectly matched layer (PML) [3], which is an absorbing region, was invented. Especially, since this time, high-order local NRBCs for the wave equation have been developed intensively (see, e.g., [4,11,21,14,15]). The exact ABCs also have been investigated for the heat equation in [17,23,24], for the advection-diffusion equation (see, e.g., [5,16]) and for the Schrödinger equation in [1,9]. Recently, Guddati et al. [12,13] use special

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absorbing boundary conditions called continued fraction absorbing boundary conditions, which are highly effective boundary conditions, for modeling wave absorption into unbounded domains. They are developed for straight and convex polygonal domains.

It is important to remark that all the above ABCs are constructed for the problems, where the right-hand side function and the initial conditions are assumed to have compact support in space.

In [22] for an elliptic problem in a semistrip a three-point vector difference scheme on an infinite interval is considered and for treating it, a method of truncation is proposed. It should be noticed that the realization of this method requires the solution of a very cumbersome auxiliary vector problem.

Differently from the above methods we approach to problems in unbounded domains by an infinite system of equations [18]. Some initial results for a stationary problem of air pollution [6,7], and several one-dimensional nonstationary problems [8] are published recently. This work is a further development of this approach for an elliptic problem in a semistrip:

$$
\begin{aligned}
&Lu=\gamma\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^2 u}{\partial y^2}+a(x)\frac{\partial u}{\partial x}-b(x)u(x,y)=f(x,y),\quad x>0,\ 0<y<1,\\
&u(x,0)=\varphi_1(x),\qquad u(x,1)=\varphi_2(x),\qquad u(0,y)=\psi(y),\qquad u(x,y)\to 0,\quad x\to+\infty,
\end{aligned}
\tag{1}
$$

under the usual assumptions that the functions in (1) are continuous and

$$
\begin{aligned}
&\gamma>0,\qquad \left|a(x)\right|\le r,\qquad b(x)\ge 0,\\
&f(x,y)\to 0,\qquad \varphi_i(x)\to 0,\quad x\to+\infty.
\end{aligned}
$$

This problem is a model of the stationary two-dimensional problem of mass propagation in a semistrip when the flow is in the x-direction. It should be emphasized that here the source function and the boundary conditions are not assumed to be of compact support.

After discretization of the problem by the difference method, using the idea of Polozhii in the method of summary representations [19] we transform the infinite system of three-point vector difference equations to infinite systems of three-point scalar equations and obtain approximate solutions with a given accuracy. Numerical experiments for several examples show the effectiveness of the proposed method.

2. Construction of difference scheme

In order to solve the problem (1) we introduce on $\bar\omega=\{x\ge 0,\ 0\le y\le 1\}$ the uniform grid

$$\bar\omega_h=\left\{(x_i,y_j),\ x_i=ih_1,\ y_j=jh_2,\ i=0,1,\dots,\ j=0,1,\dots,M\right\}.$$

Denote the boundary of $\bar\omega_h$ by

$$\Gamma_h=\left\{(x_i,0),\ (x_i,1),\ (0,y_j),\ (x_k,y_j),\ i=0,1,\dots,\ j=0,1,\dots,M,\ k\to+\infty\right\},$$

and the set of interior points by ω_h, $h=(h_1,h_2)$.

In sequel we shall use the Samarski technique and notations in [20]. Set

$$L_xu=\gamma\frac{\partial^2 u}{\partial x^2}+a(x)\frac{\partial u}{\partial x}-b(x)u(x,y)$$

and consider the associated perturbed operator

$$\widetilde{L}_xu=\varkappa\gamma\frac{\partial^2 u}{\partial x^2}+a(x)\frac{\partial u}{\partial x}-b(x)u(x,y)$$

where $\varkappa=\frac{1}{1+R}$, $R=\frac{1}{2}\frac{h_1.|a(x)|}{\gamma}$. Obviously $\varkappa$ depends only on x.

Now represent the function $a(x)$ as a sum of a nonnegative and a nonpositive terms

$$a=a^++a^-,\qquad a^+=\frac{1}{2}\big(a+|a|\big)\ge 0,\qquad a^-=\frac{1}{2}\big(a-|a|\big)\le 0.$$

Denote by $v_{i,j}$ the approximation of the values $u(x_i,y_j)$ on the grid $\bar\omega_h$,

$$\varkappa_i=\varkappa(x_i),\qquad a_i^+=a^+(x_i),\qquad a_i^-=a^-(x_i),\qquad b_i=b(x_i),\qquad f_{ij}=f(x_i,y_j),\quad (x_i,y_j)\in\bar\omega_h.$$

Next, we approximate the operator $\widetilde{L}_xu$ by the difference operator

$$\widehat{L}_xv\equiv\varkappa\gamma v_{\bar{x}x}+a^+v_x+a^-v_{\bar{x}}-bv,$$

where