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Second order scheme for scalar conservation laws with discontinuous flux

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ABSTRACT

Burger, Karlsen, Torres and Towers in [9] proposed a flux TVD (FTVD) second order scheme with Engquist–Osher flux, by using a new nonlocal limiter algorithm for scalar conservation laws with discontinuous flux modeling clarifier thickener units. In this work we show that their idea can be used to construct FTVD second order scheme for general fluxes like Godunov, Engquist–Osher, Lax–Friedrich, ... satisfying (A, B)-interface entropy condition for a scalar conservation law with discontinuous flux with proper modification at the interface. Also corresponding convergence analysis is shown. We show further from numerical experiments that solutions obtained from these schemes are comparable with the second order schemes obtained from the minimod limiter.

1. Introduction

In numerous models arising in engineering applications and applied sciences can be described by a conservation law with discontinuous flux, in particular two-phase flow problem in porous media, continuously operated clarifier thickener units and modeling traffic flow with abruptly changing road surface conditions, etc. We are interested in the following single conservation law in one space dimension,

$$\begin{aligned} &u_t + F\big(k(x), u\big)_x = 0, \\ &u(x, 0) = u_0(x) \end{aligned} \tag{1}$$

where the flux function F depends on the space variable through a coefficient k which may be discontinuous. For simplicity we consider that the flux function has only single discontinuity at the point $x = 0$. In this case the flux function F is of the form

$$F\big(k(x), u\big) = k(x) f(u) + \big(1 - k(x)\big) g(u) \tag{2}$$

where f and g are Lipschitz continuous functions on the interval $I = [s, S]$ and $k(x)$ is the Heaviside function. We assume that the flux functions f, g satisfies the following hypothesis,

(H_1) $f(s) = g(s)$, $f(S) = g(S)$,
(H_2) f and g have one global maximum and no other local maximum in $[s, S]$, see Fig. 1.

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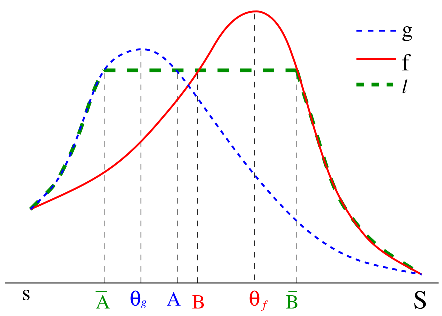

Fig. 1. l function for (A, B)-connection.

Denote by $Lip(f)$ and $Lip(g)$ the Lipschitz constants of f and g. Also we define the constant

$$M = \max\{Lip(f), Lip(g)\}.$$

Eq. (1) has been widely studied from both the theoretical and numerical points of view in recent years. Several existence results for the entropy solutions have been obtained by using regularization of coefficients as in [12], by front tracking as in [11,15], by explicit Hopf–Lax formulas in [7] and by proving convergence of numerical schemes of the Godunov or Engquist–Osher type as in [5,6,16,18,19,13,8] and the Lax–Friedrichs type as in [14]. Later in [4] a new concept of entropy solutions is introduced, namely, the Optimal entropy solutions. This is based on a two-step approach: first fix an interface connection ((A, B)-connection) and next define an interface entropy condition with respect to this connection, called (A, B)-interface entropy condition. The corresponding (A, B) entropy solutions were shown to be L^1 contractive for every choice of the interface connection. Existence of such a solution satisfying (A, B)-entropy condition was proved by showing that a Godunov type scheme converges to that entropy solution. In [10], (A, B)-entropy condition satisfying Engquist–Osher scheme is studied. Further in [1], in general, first order monotone schemes satisfying (A, B)-entropy conditions are studied. These schemes need not be total variation bounded (see [2]) but it satisfy flux TVD property. Now the question is how to construct a second order flux TVD scheme. In [9], introduced a nonlocal limiter algorithm to construct Engquist–Osher type second order schemes with flux TVD property for conservation laws with discontinuous flux modeling clarifier thickener units. In this work, we show, their idea can be extended to construct second order schemes satisfying (A, B)-interface entropy condition with flux TVD property not only to Engquist–Osher but also to other fluxes like Godunov, Lax–Friedrichs, ... Numerical experiments shows the existence of a dog-leg feature in the approximated solution which is not reducing as mesh size reduces (see Section 5). To overcome this difficulty, in the sweeping algorithm we add Ch^{δ} for suitable C large and $\delta \in (0, 1)$. Also adding Ch^{δ} in the sweeping algorithm is needed to show the approximate solution satisfies the (A, B)-interface entropy condition.

The paper is organized as follows. In Section 2 we give some basic definitions and results. In Section 3 first order monotone scheme is briefly explained. In Section 4 construction of second order resolution numerical scheme is given and the main theorem is stated. Through Section 5 to Section 7 we establish the convergence of the numerical scheme to an (A, B)-entropy solution of (1). In Section 8 numerical results are presented.

2. Basic definitions and results

In this section we give some known definitions and results which will be used later.

Definition. Weak solution: $u \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ is said to be a weak solution of Eq. (1) if $\forall \phi \in C_0^{\infty}(\mathbb{R} \times \bar{\mathbb{R}}_+)$, we have that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \big(u\phi_t + F\big(k(x), u\big)\phi_x\big)\,dx\,dt + \int_{\mathbb{R}} \phi(x, 0)u_0(x)\,dx = 0. \tag{3}$$

Now we define the entropy conditions that hold away from the interface $x = 0$.

Definition. Entropy flux pairs: for $i = 1, 2$, (φ_i, ψ_i) are said to be entropy pairs if φ_i is a convex function on $[s, S]$ and $(\psi_1'(\theta), \psi_2'(\theta)) = (\varphi_1'(\theta) f'(\theta), \varphi_2'(\theta) g'(\theta))$ $\forall \theta \in [s, S]$.

Definition. Interior entropy condition: A function $u \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ is said to satisfy the interior entropy condition if it satisfies

$$\frac{\partial}{\partial t}\varphi_1(u) + \frac{\partial}{\partial x}\psi_1(u) \leqslant 0 \quad \text{in } x > 0, t > 0,$$