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The interior inverse scattering problem for cavities with an artificial obstacle

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ABSTRACT

The interior inverse scattering by an impenetrable cavity is considered. Both the sources and the measurements are placed on a curve or surface inside the cavity. As a rule of thumb, both the direct and the inverse problems suffer from interior eigenvalues. The interior eigenvalues are removed by adding an artificial obstacle with impedance boundary condition to the underlying scattering system. For this new system, we prove a reciprocity relation for the scattered field and a uniqueness theorem for the inverse problem. Some new techniques are used in the arguments of the uniqueness proof because of the Lipschitz regularity of the boundary of the cavity. The linear sampling method is used for this new scattering system for reconstructing the shape of the cavity. Finally, some numerical experiments are presented to demonstrate the feasibility and effectiveness of the linear sampling method. In particular, the introduction of the artificial obstacle makes the linear sampling method robust to frequency.

1. Introduction

The *interior* inverse scattering problems are fairly new research topics for reconstructing the shape of a cavity from the measurements taken on a curve or surface inside the cavity [10,11,17–23]. These problems occur in many industrial applications of non-destructive testing where both the sources (incident waves) and measurements (scattered waves) are inside the cavity [11,19]. As noted in [19], in some ways the interior inverse scattering problem is physically more complicated since the scattered waves are "trapped" inside the cavity.

Consider an impenetrable cavity $D \subset \mathbb{R}^2$, which is assumed to be a bounded simply connected domain with Lipschitz boundary ∂D. Let $k = \omega/c > 0$ be the wave number where $\omega > 0$ denotes the frequency of a time harmonic wave and $c > 0$ the sound speed. Let C be a closed curve inside the cavity and D_0 be the interior domain of C. Throughout, we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_0. Note that this requirement is not essential since we have the freedom to choose C. The incident fields u^i are taken to be point sources of the form

$$u^i(x) = \Phi(x,z) := \frac{i}{4} H_0^{(1)}\big(k|x-z|\big), \quad x \in \mathbb{R}^2,\ z \in C, \tag{1.1}$$

where $H_0^{(1)}$ is the *Hankel function* of the first kind of order zero. The scattered field u^s satisfies the Helmholtz equation

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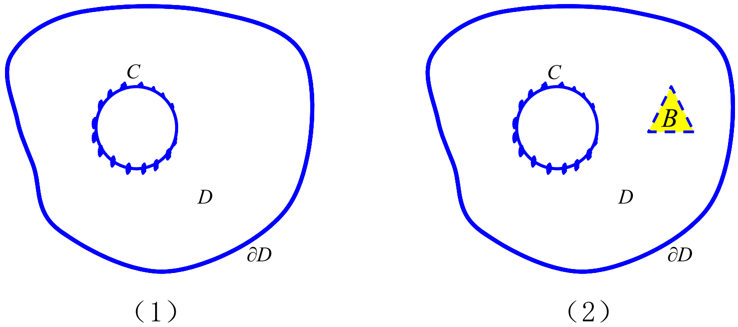

Fig. 1. Cavity D and the curve C where measurements are taken. (1): The typical scattering system; (2) The new scattering system with an artificial obstacle B.

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } D. \tag{1.2}$$

For convenience, we will consider the following general mixed boundary conditions for the total field $u = u^i + u^s$:

$$u = 0 \quad \text{on } \partial D_D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D_N, \tag{1.3}$$

where ∂D_D and ∂D_N form a Lipschitz dissection of ∂D, ν denotes the exterior unit normal vector at $x \in \partial D$. This general formulation covers the Dirichlet boundary condition ($\partial D_N = \emptyset$) and the Neumann boundary condition ($\partial D_D = \emptyset$). We did not study the impedance boundary conditions because no interior eigenvalue occurs in this case. In the following, we will denote the mixed boundary condition on ∂D (1.3) by $\mathscr{B}(u) = 0$.

Under the following Assumption 1.1, the well-posedness of the scattering problem (1.2)–(1.3) can be proved in a standard way as in Example 5.15 in [2] for the case of only Dirichlet boundary condition. We also refer to Theorem 8.4 and Remark 8.6 in [2] for the mixed boundary condition (1.3). Here we remark that as apposed to the exterior scattering problem, the presence of the interior eigenvalues will destroy the uniqueness of the direct interior scattering problem (1.2)–(1.3). The same barrier also occurs in the inverse scattering problems. Physically, if the cavity is a perfect reflector, probing with a source at an eigenfrequency will resolve to a resonance state.

Assumption 1.1. k^2 is NOT an interior eigenvalue of $-\Delta$ with respect to the boundary condition (1.3) under consideration.

The inverse scattering problem is to determine the shape of the cavity D from the scattered fields $u^s(x,z)$ for all $x \in C$ due to point sources $u^i(\cdot,z)$ for all $z \in C$ (see Fig. 1(1)).

The uniqueness of the inverse problem has been established in [17,19,23] for the Dirichlet boundary condition, in [17,20] for the impedance boundary condition and in [10] for the mixed boundary condition. Here we note that these uniqueness results are proved under Assumption 1.1. To the author's knowledge it is still an open problem whether the scattered fields $u^s(x,z)$ for all $x \in C$ but only one incident wave $u^i(\cdot,z)$ uniquely determines D without any additional a priori information. Partial results are known if a priori information on the cavity D is available [17,18]. Recently, Hu and Liu [9] proved that a single point source is sufficient to uniquely determine D provided we known D is a polyhedron or a ball in advance.

Some numerical reconstruction methods have been proposed to solve this kind of inverse scattering problems. Assuming the measurements $u^s(x,z)$ are taken for all $x, z \in C$, for the Dirichlet and impedance cases, the linear sampling method has been applied by Qin and Colton [19,20] and the factorization method has been studied by Liu [17]. Zeng, Cakoni and Sun [22] studied the linear sampling method for inverse electromagnetic scattering by a perfect conducting cavity. Using the scattered field u^s on C for only one point source, we refer to a regularized Newton iterative method by Qin and Cakoni [18], Qin and Liu [21], and a decomposition method by Zeng, Suarez and Sun [23]. The linear sampling method has also been applied for the case of mixed typed boundary conditions by Hu, Cakoni and Liu [10]. The same as for the well-posedness of the direct problem and the uniqueness of the inverse problem, these numerical methods are employed under Assumption 1.1. Actually, the final numerical example given in this paper shows that the reconstructions by using the classical linear sampling method get worse when k^2 is close to an interior eigenvalue.

To remove Assumption 1.1, in this paper, we introduce an artificial obstacle $B \subset \mathbb{R}^2$ of Lipschitz class such that $\overline{B} \subset D$ and $D \backslash \overline{B}$ is connected (see Fig. 1(2)). Note that more than one (but finitely many) component obstacle is allowed. We impose the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + i\lambda_0 u = 0 \quad \text{on } \partial B, \tag{1.4}$$