Applied Numerical Mathematics 88 (2015) 46–65

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

# Superconvergence of discontinuous Galerkin solutions for higher-order ordinary differential equations

H. Temimi

*Department of Mathematics & Natural Sciences, Gulf University for Science & Technology, P.O. Box 7207, Hawally 32093, Kuwait*

ARTICLE INFO

*Article history:*
Received 11 February 2014
Received in revised form 18 June 2014
Accepted 15 September 2014
Available online 30 October 2014

*Keywords:*
Discontinuous
Galerkin
Superconvergence
Higher-order
A *posteriori*
Error
Estimates

ABSTRACT

In this paper, we study the superconvergence properties of the discontinuous Galerkin (DG) method applied to one-dimensional $m$th-order ordinary differential equations without introducing auxiliary variables. We show that the leading term of the discretization error on each element is proportional to a combination of Jacobi polynomials. Thus, the $p$-degree DG solution is $O(h^{p+2})$ superconvergent at the roots of specific combined Jacobi polynomials. Moreover, we use these results to compute simple, efficient and asymptotically exact a *posteriori* error estimates and to construct higher-order DG approximations.


## 1. Introduction

The discontinuous Galerkin method considered here is a class of finite element method using completely discontinuous piecewise polynomials for the numerical solution and the test functions. The DG method was highly implemented to provide computational solutions of several partial differential equations. In addition to their finite element nature, the DG methods are stable, locally conservative and easy to implement, DG methods do not require continuity across element boundaries and can handle complex geometry and unstructured meshes with hanging nodes. Moreover, DG methods simplify adaptive $h$–$p$ refinement and produce efficient parallel solution procedures.

In 1973, the DG method was first developed by Reed and Hill [13] to solve first order steady state linear hyperbolic conservation laws. Later, Cockburn and Shu [10] extended the method to solve first-order hyperbolic partial differential equations of conservation laws. They also developed the Local Discontinuous Galerkin (LDG) method for convection–diffusion problems [11]. However, the DG method left asleep for many decades till the last twenty years when it started attracting the attention of several researchers and gained much more popularity due to its wide application and flexibility.

Recently, the superconvergence properties of the DG method have been intensively analyzed, these properties can be used to construct efficient and asymptotically correct a *posteriori* estimates of the discretization errors. Adjerid et al. [2] showed that DG solutions of first-order differential equations are $O(h^{p+2})$ superconvergent at Radau points and exhibit a strong $O(h^{2p+1})$ superconvergence at the downwind nodes of each element. Celiker and Cockburn [8] showed that the $p$-degree DG solution and its derivative are, respectively, $O(h^{p+2})$ and $O(h^{p+1})$ superconvergent at the $p$-degree right Radau and at $p$-degree left Radau polynomials. Meng et al. [12] provided superconvergence results of the LDG method applied to one dimensional linear time dependent fourth order problems, they proved that the error achieves $(p+\frac{3}{2})$th

*E-mail address:* temimi.h@gust.edu.kw.

http://dx.doi.org/10.1016/j.apnum.2014.09.009
0168-9274/

order of convergence between the LDG solution and a particular projection of the exact solution when using polynomial of degree $p$. Baccouch applied the LDG method to the fourth-order Euler–Bernoulli partial differential equation in one space dimension, he [5] developed and analyzed a new superconvergent LDG method and proved the $\mathcal{L}^2$ stability of the scheme and several optimal $\mathcal{L}^2$ error estimates for the solution and its spatial derivatives (up to third order), he [6], also, showed that the significant parts of the discretization errors for the LDG solution and its spatial derivatives (up to third order) are proportional to $(p+1)$-degree Radau polynomials, when polynomials of total degree not exceeding $p$ are used.

Yang and Shu [15] applied the LDG to one-dimensional linear parabolic equation, they proved that the error between the LDG solution and the exact solution is $(p+2)$th order superconvergent at the Radau points with suitable initial discretization and they proved that the LDG solution is $(p+2)$th order superconvergent for the error to a particular projection of the exact solution when using piecewise $p$th degree polynomials. Baccouch analyzed the superconvergence properties of the LDG method applied to the second-order wave equation in one space dimension, he [4] showed that the LDG solution is $O(h^{p+2})$ superconvergent at the $(p+1)$-degree right-Radau polynomial and the solution's derivative is $O(h^{p+2})$ superconvergent at the $(p+1)$-degree left-Radau polynomial; and he [7] proved that the LDG solution and its spatial derivative are $O(h^{p+\frac{3}{2}})$ super close to particular projections of the exact solutions for $p$th-degree polynomial spaces.

Unfortunately, researchers were not fully satisfied with the LDG method due to the introduction of new auxiliary variables and transformation of the original equation into a system of several first order equations which leads to a more complex DG method with expensive computational cost. Unlike the LDG method, Adjerid and Temimi [3] introduced a DG method for solving higher-order initial value problem without introducing auxiliary variables. They performed double integration by parts and chose downwind numerical fluxes for the solution and its derivatives. They showed that the *p-degree* DG solution is $O(h^{p+2})$ superconvergent at the roots of the $(p+1-m)$-*degree* Jacobi polynomial $P^{m,0}_{p+1-m}(\tau)$ and they showed that the *p-degree* DG solution and its first $m-1$ derivatives are $O(h^{2p+2-m})$ superconvergent at the end of each step where $m$ is the order of the ordinary differential equation.

Cheng and Shu [9] presented an alternative DG method for solving time dependent partial differential equations with higher-order spatial derivatives. Similar to [3], this method does not require the use of auxiliary variables but relies on careful design of the numerical fluxes in order to ensure the stability of the method. Moreover, they proved $(p+1)$th order of accuracy when using piecewise $p$th degree polynomials, under the condition that $p+1$ is greater than or equal to the order of the equation. These DG schemes are more compact and simpler in formulation the classical LDG method. Stimulated by the work of Cheng and Shu [9] and in order to develop more properties of the DG method, a local error analysis was conducted in this manuscript, we show that the leading term of the discretization error on each element is proportional to a combination of Jacobi polynomials. Thus, the $p$-degree DG solution is $O(h^{p+2})$ superconvergent at the roots of specific combined Jacobi polynomials. These results are used to compute simple, efficient and asymptotically exact a *posteriori* error estimates and to construct higher-order DG approximations.

The paper is organized as follows: In Section 2, we present the DG formulation for second-order ordinary differential equations. In Section 3, we provide a local error analysis of the DG method. We extend the error analysis of the DG method to $m$th-order ordinary differential equations in Section 4 and to nonlinear ordinary differential equations in Section 5. In Section 6, we construct efficient and asymptotically exact a *posteriori* error estimates. In Section 7, we present several numerical results to show the full agreement with the theory. In Section 8, we conclude with a few remarks.

## 2. A model problem

Let us consider the following second-order ordinary boundary value problem

$$c_2 u'' + c_1 u' + c_0 u = f(x), \quad a < x < b, \tag{2.1a}$$

subject to the mixed boundary conditions

$$u(a) = u_l, \qquad u'(b) = u_{rx}, \tag{2.1b}$$

we will also explore numerically (2.1a) subject to the Dirichlet boundary conditions

$$u(a) = u_l, \qquad u(b) = u_r. \tag{2.2}$$

Assume that $f$ is selected such that the exact solution is a smooth function.

In order to obtain the weak DG formulation, we partition the interval $[a,b]$ into $N+1$ subintervals $I_k = (x_k, x_{k+1})$, $k = 0, \cdots, N$, where $x_0 = a$, $x_k = a + k\,\Delta x$, $k = 1, 2, \cdots, N$ and $x_{N+1} = b$, we denote by $\Delta x = \frac{(b-a)}{N+1}$ the length of each subinterval.

We define a finite element space consisting of piecewise $p$th-degree polynomial functions

$$S^{N,p} = \{U \,:\, U|_{I_k} \in \mathcal{P}_p\}, \tag{2.3}$$

where $\mathcal{P}_p$ denotes the space of polynomials of degree $p$. We define the weak DG formulation for (2.1) by multiplying (2.1a) by a test function, and then integrating over $I_k$. A double integration by parts leads to