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Analysis of errors in some recent numerical quadrature formulas for periodic singular and hypersingular integrals via regularization

Avram Sidi

Computer Science Department, Technion – Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

Recently, we derived some new numerical quadrature formulas of trapezoidal rule type for the singular integrals $I^{(1)}[u] = \int_a^b (\cot \frac{\pi(x-t)}{T}) u(x)\,dx$ and $I^{(2)}[u] = \int_a^b (\csc^2 \frac{\pi(x-t)}{T}) u(x)\,dx$, with $b-a=T$ and $u(x)$ a T-periodic continuous function on $\mathbb{R}$. These integrals are not defined in the regular sense, but are defined in the sense of Cauchy Principal Value and Hadamard Finite Part, respectively. With $h=(b-a)/n$, $n=1,2,\ldots,$ the numerical quadrature formulas $Q_n^{(1)}[u]$ for $I^{(1)}[u]$ and $Q_n^{(2)}[u]$ for $I^{(2)}[u]$ are

$$Q_n^{(1)}[u] = h\sum_{j=1}^{n} f(t+jh-h/2), \qquad f(x) = \left(\cot\frac{\pi(x-t)}{T}\right)u(x),$$

and

$$Q_n^{(2)}[u] = h\sum_{j=1}^{n} f(t+jh-h/2) - T^2u(t)h^{-1}, \qquad f(x) = \left(\csc^2\frac{\pi(x-t)}{T}\right)u(x).$$

We provided a complete analysis of the errors in these formulas under the assumption that $u \in C^\infty(\mathbb{R})$ and is T-periodic. We actually showed that,

$$I^{(1)}[u] - Q_n^{(1)}[u] = O\left(n^{-\mu}\right) \quad \text{and}$$
$$I^{(2)}[u] - Q_n^{(2)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n\to\infty,\ \forall \mu > 0.$$

In this note, we analyze the errors in these formulas under the weaker assumption that $u \in C^s(\mathbb{R})$ for some finite integer s. By first regularizing these integrals, we prove that, if $u^{(s+1)}$ is piecewise continuous, then

$$I^{(1)}[u] - Q_n^{(1)}[u] = o\left(n^{-s-1/2}\right) \quad \text{as } n\to\infty,\ \text{if } s \geqslant 1, \quad \text{and}$$
$$I^{(2)}[u] - Q_n^{(2)}[u] = o\left(n^{-s+1/2}\right) \quad \text{as } n\to\infty,\ \text{if } s \geqslant 2.$$

We also extend these results by imposing different smoothness conditions on $u^{(s+1)}$. Finally, we append suitable numerical examples.

E-mail address: asidi@cs.technion.ac.il.
URL: http://www.cs.technion.ac.il/~asidi.

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1. Introduction and background

Let $u(x)$ be a T-periodic function that is sufficiently smooth on $\mathbb{R}$ and consider the singular integrals

$$I^{(1)}[u] = \int_a^b \left(\cot\frac{\pi(x-t)}{T}\right) u(x)\,dx, \quad a < t < b;\ b - a = T, \tag{1.1}$$

and

$$I^{(2)}[u] = \int_a^b \left(\csc^2\frac{\pi(x-t)}{T}\right) u(x)\,dx, \quad a < t < b;\ b - a = T. \tag{1.2}$$

Of these, $I^{(1)}[u]$ is known also as the *circular Hilbert transform* and is defined in the sense of *Cauchy Principal Value (CPV)*, while $I^{(2)}[u]$ is a so-called *hypersingular integral* and is defined in the sense of *Hadamard Finite Part (HFP)*. Note that the integrands in both integrals are T-periodic with nonintegrable singularities at $x = t$ in the interval of integration (a, b); namely, the integrand of $I^{(1)}[u]$ has a singularity of the form $(x-t)^{-1}$, while $I^{(2)}[u]$ has a singularity of the form $(x-t)^{-2}$. For the properties of CPV and HFP integrals, see Davis and Rabinowitz [3], Evans [4], or Kythe and Schäferkotter [5], for example.[1]

In the recent papers Sidi and Israeli [14] and Sidi [12], we derived trapezoidal rule type approximations to the integrals $I^{(1)}[u]$ and $I^{(2)}[u]$, respectively.[2] With $h = T/n$, $n = 1, 2, \dots,$ these approximations are

$$Q_n^{(1)}[u] = h\sum_{j=1}^{n} f(t + jh - h/2), \qquad f(x) = \left(\cot\frac{\pi(x-t)}{T}\right)u(x), \quad \text{for } I^{(1)}[u], \tag{1.3}$$

and

$$Q_n^{(2)}[u] = h\sum_{j=1}^{n} f(t + jh - h/2) - T^2 u(t) h^{-1}, \qquad f(x) = \left(\csc^2\frac{\pi(x-t)}{T}\right)u(x), \quad \text{for } I^{(2)}[u]. \tag{1.4}$$

We also derived the following results concerning the errors in these approximations under the assumption that $u \in C^\infty(\mathbb{R})$:

$$Q_n^{(1)}[u] - I^{(1)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n \to \infty,\ \forall \mu > 0, \tag{1.5}$$

and

$$Q_n^{(2)}[u] - I^{(2)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n \to \infty,\ \forall \mu > 0. \tag{1.6}$$

As is done in [12], both of these results can be derived by using one of the author's generalizations of the classical Euler–Maclaurin expansion given in Sidi [10, Theorem 2.3].[3]

In this work, we analyze the errors in the formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ under the weaker assumption that $u(x)$ is

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[1] The usual notation for integrals defined in the sense of the Cauchy Principal Value (CPV) is $⨍_a^b f(x)\,dx$, while for those defined in the sense of Hadamard Finite Part (HFP) it is $⨎_a^b f(x)\,dx$. For simplicity, in this work, we use $\int_a^b f(x)\,dx$ to denote both, as in (1.1) and (1.2).

[2] Actually, in [12], we treated singular integrals of the very general forms

$$I[g] = \int_a^b g(x)\bigl(\log|x-t|\bigr)^p |x-t|^\beta, \quad \beta \text{ real},\ p = 0, 1,\ a < t < b,$$

and

$$I[g] = \int_a^b g(x)(x-t)^\beta, \quad \beta = -1, -2, \dots,\ a < t < b,$$

$g(x)$ being allowed to have arbitrary algebraic endpoint singularities. Of course, $I^{(1)}[u]$ and $I^{(2)}[u]$ are special cases of these.

[3] The classical Euler–Maclaurin (E–M) expansion pertains to integrals $\int_a^b g(x)\,dx$ whose integrands are regular throughout the (closed) interval $[a, b]$, whereas the generalized E–M expansions of [10] treat the case in which $g(x)$ has arbitrary algebraic endpoint singularities. For the case in which $g(x)$ has arbitrary algebraic-logarithmic endpoint singularities, see Sidi [9] and [11]. In all three papers [9], [10], and [11], the integrals $\int_a^b g(x)\,dx$ can be convergent or divergent; in case of divergence, they are defined in the sense of HFP.

It was observed in [12] that the CPV integrals are also sums of HFP integrals, and this fact was used in the derivation of their associated asymptotic expansions.