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Analytic and numerical exponential asymptotic stability of nonlinear impulsive differential equations

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ABSTRACT

This paper deals with exponential stability of both analytic and numerical solutions to nonlinear impulsive differential equations. Instead of Lyapunov functions a new technique is used in the analysis. A sufficient condition is given under which the analytic solution is exponential asymptotically stable. The numerical solutions are calculated by Runge–Kutta methods and the corresponding stability properties are studied. It is proved that algebraically stable Runge–Kutta methods satisfying $|1 - b^T A^{-1} e| < 1$ can preserve the stability of the equation. Finally some numerical experiments are given to illustrate the conclusion.

1. Introduction

The impulsive differential equations which enrich the theory of differential equations are widely used in actual modeling such as population dynamics, epidemic, optimal control, etc. For instance, in [12,14,15], the models are constructed by impulsive differential equations.

Up to now extensive work has been done in the area of impulsive differential equations. In [1,5,6,8], the authors studied the properties of the analytic solutions. In [11], the authors considered the numerical solutions of linear scalar autonomous impulsive differential equations. The result is surprising because it has been proved that the explicit Euler method is stable for the linear impulsive differential equations while the implicit Euler method isn't. Hence the study of numerical solutions of impulsive differential equations is not an extension of ordinary differential equations. Since then researchers began to interest in the numerical properties (see [3,7,10]). In [9], the authors investigated the convergence and stability of an improved linear multistep methods to the same linear equations. In [7] and [10], Liang and Liu studied asymptotical stability of Runge–Kutta methods to multi-dimensional linear impulsive differential equations. Ding considered the convergence of Euler methods for linear impulsive delay differential equations in [3]. Above all, previous literature focused on the numerical properties of linear impulsive differential equations and there are still few papers on Runge–Kutta methods for nonlinear impulsive differential equations.

In this paper, we consider the following equation:

$$\begin{cases} x'(t) = f\big(t, x(t)\big), & t \geqslant 0,\ t \neq k, \\ \Delta x = rx(t), & t = k,\ k \in N, \\ x\big(0^+\big) = x_0, \end{cases} \tag{1.1}$$

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where $f : [0, +\infty) \times \mathbb{C}^q \to \mathbb{C}^q$ $r \in \mathbb{C}$ and $r \neq -1$. We assume $\triangle x = x(t+0) - x(t)$, where $x(t+0)$ is the right limit of $x(t)$.

Remark 1.1. In the case that $r = -1$, we have $x(1+) = 0$ for arbitrary initial value, i.e., all solutions of (1.1) are equal to each other for $t > 1$. Therefore we omit this case in our paper.

Definition 1.2. (See [2].) A function $x(t) \in \mathbb{C}^q$ is said to be the solution of (1.1), if

(1) $\lim_{t\to 0^+} x(t) = x_0$,
(2) For $t \in (0, +\infty)$, $t \neq k$, $k \in N$, $x(t)$ is differentiable and $x'(t) = f(t, x(t))$,
(3) $x(t)$ is left continuous in $(0, +\infty)$ and $x(k^+) = (1+r)x(k)$, $k \in N$.

In order to investigate the stability of $x(t)$, we also consider the following equation:

$$\begin{cases} \bar{x}'(t) = f\bigl(t, \bar{x}(t)\bigr), & t \geqslant 0,\ t \neq k, \\ \Delta\bar{x} = r\bar{x}(t), & t = k, \\ \bar{x}\bigl(0^+\bigr) = \bar{x}_0. \end{cases} \tag{1.2}$$

We assume that (1.1) has a unique solution in this paper. The rest of the paper is organized as follows. In Section 2, we study the stability of analytic solution to (1.1). In Sections 3 and 4, we try to find the Runge–Kutta methods that can preserve the property. In Section 5, we consider the equation in linear scalar case and give a comparison with the results in [11]. At last, numerical experiments are given to illustrate the conclusion in the paper.

2. Stability analysis of the analytic solution

In this section, we will give the stability condition of the analytic solution to nonlinear impulsive differential equations.

Definition 2.1. The solution $x(t)$ to Eq. (1.1) is called stable, if for arbitrary x_0 and $\bar{x}_0$, there exist a constant d_0 such that

$$\bigl\|x(t) - \bar{x}(t)\bigr\| \leqslant d_0\|x_0 - \bar{x}_0\|.$$

Furthermore, the solution $x(t)$ to Eq. (1.1) is called exponential asymptotically stable, if for arbitrary x_0 and $\bar{x}_0$, there exist constants $d_1 > 0$ and $d_2 < 0$ such that

$$\bigl\|x(t) - \bar{x}(t)\bigr\| \leqslant e^{d_2 t} d_1\|x_0 - \bar{x}_0\|.$$

Denote

$$y(t) = \begin{cases} (1+r)^{\{t\}}x(t), & t \geqslant 0,\ t \neq k, \\ (1+r)x(k), & t = k, \end{cases} \tag{2.1}$$

where $\{t\}$ denotes the fractional part of t. Then it is easy to see that $y(t)$ is continuous for $t \in [0, +\infty)$. Consider the following differential equation

$$\begin{cases} y'(t) = y(t)\ln(1+r) + (1+r)^{\{t\}} f\bigl(t, (1+r)^{-\{t\}}y(t)\bigr), \\ y(0) = x_0, \end{cases} \tag{2.2}$$

for which we will define the solution in the underlying definition.

Remark 2.2. In (2.2), we define $y'(t)$ as the right-hands derivative when $t = k$, i.e.,

$$y'(k) = D_+ y(k) = \lim_{t\to k+} y'(t), \quad k = 1, 2, \ldots.$$

Definition 2.3. A solution of (2.2) on $[0, \infty)$ is a function $y(t)$ that satisfies the following conditions:

(i) $y(t)$ is continuous on $[0, \infty)$;
(ii) The derivative $y'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist;
(iii) (2.2) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end-points.

The following theorem provides a way to transform Eq. (1.1) into Eq. (2.2).