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Cartesian PML approximation to resonances in open systems in $\mathbb{R}^2$

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ABSTRACT

In this paper, we consider a Cartesian PML approximation to resonance values of time-harmonic problems posed on unbounded domains in $\mathbb{R}^2$. A PML is a fictitious layer designed to find solutions arising from wave propagation and scattering problems supplemented with an outgoing radiation condition at infinity. Solutions obtained by a PML coincide with original solutions near wave sources or scatterers while they decay exponentially as they propagate into the layer. Due to rapid decay of solutions, it is natural to truncate unbounded domains to finite regions of computational interest. In this analysis, we introduce a PML in Cartesian geometry to transform a resonance problem (characterized as an eigenvalue problem with improper eigenfunctions) on an unbounded domain to a standard eigenvalue problem on a finite computational region. Truncating unbounded domains gives rise to perturbation of resonance values, however we show that eigenvalues obtained by the truncated problem converge to resonance values as the size of computational domain increases. In addition, our analysis shows that this technique is free of spurious resonance values provided truncated domains are sufficiently large. Finally, we present the results of numerical experiments with simple model problems.

1. Introduction

In this paper we will analyze perfectly matched layer (PML) approximation based on Cartesian geometry to resonance values of problems on unbounded domains in $\mathbb{R}^2$. Research on resonances in open systems has been extensively developed because of their many potential applications. For example, applications of acoustic resonance include designing musical instruments such as violins and guitars [13,23] and determining frequencies of acoustic noise arising from an airplane wing and its slat and flap (see [21] and reference therein). The other example is photonic resonances and they take place in special structures of dielectric materials. It is known that periodic dielectric structures (photonic crystals) can prohibit waves of frequencies in a particular range (called a photonic band gap or PBG) from propagating in the structures [25,30,34]. While the ideal photonic crystals have an infinite periodic pattern, in a practical application dielectric materials are arranged in a periodic pattern to a finite extent [15,35]. If a defect is introduced in the structures, then they may produce localized resonance modes of frequencies in PBG. This unique properties of photonic crystals allow many applications including lasers, waveguides, optical filters and optical communications.

In this paper, for acoustic models we consider a problem to find complex wavenumbers k for which there exists a nonzero solution u satisfying

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$$\begin{aligned} &\Delta u + k^2 u = 0 \quad \text{in } \bar{\Omega}^c, \\ &u = 0 \quad \text{on } \Gamma \end{aligned} \tag{1.1}$$

with an *outgoing radiation condition* at infinity, which will be discussed below. Here Ω is a bounded scatterer with a Lipschitz boundary Γ and we denote the complement of the closure of Ω in $\mathbb{R}^2$ by $\bar{\Omega}^c := \mathbb{R}^2 \setminus \bar{\Omega}$.

For photonic resonance models, we consider two basic polarizations of electromagnetic equations with dielectric materials contained in a bounded region of $\mathbb{R}^2$. In case of the TE polarization, magnetic fields satisfy the scalar Helmholtz equation

$$\begin{aligned} &\nabla \cdot \frac{1}{\varepsilon} \nabla u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \partial G, \\ &u^- - u^+ = 0 \quad \text{on } \partial G, \\ &\frac{1}{\varepsilon^-} \frac{\partial u^-}{\partial n} - \frac{1}{\varepsilon^+} \frac{\partial u^+}{\partial n} = 0 \quad \text{on } \partial G, \end{aligned} \tag{1.2}$$

where G is a finite-sized periodic dielectric material and ε is a dielectric constant of the photonic structure such that $\varepsilon = \varepsilon^+ = 1$ on the background material and $\varepsilon = \varepsilon^-$ on G. Also, u^+ and u^- represent the restriction of the function u to $\mathbb{R}^2 \setminus \bar{G}$ and G respectively, and in the transmission conditions on ∂G, $u^\pm$ and $\partial u^\pm / \partial n$ are understood as their traces on ∂G with n the outward unit normal vector on the boundary of G.

For the TM polarization, electric fields satisfy

$$\begin{aligned} &\Delta u + k^2 \varepsilon u = 0 \quad \text{in } \mathbb{R}^2 \setminus \partial G, \\ &u^- - u^+ = 0 \quad \text{on } \partial G, \\ &\frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = 0 \quad \text{on } \partial G. \end{aligned} \tag{1.3}$$

As in the acoustic model problem (1.1), the model problems (1.2) and (1.3) require an outgoing radiation condition at infinity. In this paper, for $|\arg(k)| < \pi$ a solution $u \in H^1_{loc}(\bar{\Omega}^c)$ to the Helmholtz equation is said to be an *outgoing* solution if u has a series representation in terms of Hankel functions of the first kind

$$u(x) = \sum_{n=-\infty}^{\infty} a_n H_n^1\big(k|x|\big) e^{in\theta_x} \quad \text{for } |x| > r_0 \tag{1.4}$$

for some $r_0 > 0$, where $\theta_x = \arg(x)$ and H_n^1 are Hankel functions of the first kind of order n [1,32]. Now, we are interested in k for which the model problems have nonzero solutions and such a k is called a *resonance*.

In the acoustic scattering theory, it is known that for k with $\mathrm{Im}(k) \geqslant 0$, the outgoing radiation condition given by a series (1.4) is equivalent to the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \tag{1.5}$$

with $r = |x|$ (see e.g, [11]). Furthermore, by using variational arguments one can show that the Helmholtz equation with $\mathrm{Im}(k) \geqslant 0$ supplemented with the Sommerfeld radiation condition (1.5) has a unique solution [10]. Therefore resonance values have necessarily a negative imaginary part. Due to this fact and the outgoing radiation condition (1.4) together with an asymptotic behavior of Hankel functions of the first kind (3.8), one can show that resonance functions are not square integrable. Hence they can be thought of as *improper* eigenfunctions.

A PML is an artificial absorbing layer surrounding the area of computational interest. This fictitious layer can be introduced by a certain complex coordinate stretching in a way that solutions obtained by the method are preserved outside of PML and decay exponentially in the layer. So it is natural to truncate unbounded domains to a finite region, which allows one to apply standard computational techniques, e.g., finite element methods. Since Bérenger proposed a PML method to study electromagnetic waves [4,5] in time domain, many accurate and efficient variants of PML were applied to many different areas such as acoustics [6,31], elastics [8,19,18] and electromagnetics [6,7,9] in time domain and frequency domain.

Also, PML methods have successfully employed for computing acoustic resonances [21,20] and photonic resonances [17, 24]. A PML technique deforms the original resonance problem to a standard eigenvalue problem posed on a bounded domain in the following steps

(i) resonance problem (problem with improper eigenfunctions),
(ii) eigenvalue problem posed in an unbounded domain (infinite PML problem),
(iii) eigenvalue problem posed in a bounded domain (truncated PML problem).