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A flux preserving immersed nonconforming finite element method for elliptic problems

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ABSTRACT

An immersed nonconforming finite element method based on the flux continuity on intercell boundaries is introduced. The direct application of flux continuity across the support of basis functions yields a nonsymmetric stiffness system for interface elements. To overcome non-symmetry of the stiffness system we introduce a modification based on the Riesz representation and a local postprocessing to recover local fluxes. This approach yields a P_1 immersed nonconforming finite element method with a slightly different source term from the standard nonconforming finite element method. The recovered numerical flux conserves total flux in arbitrary sub-domain. An optimal rate of convergence in the energy norm is obtained and numerical examples are provided to confirm our analysis.
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1. Introduction

In this paper, we consider a simple model interface problem:

$$-div\,(\kappa\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where the domain $\Omega = \Omega_- \cup \Omega_+$ is a simply connected, bounded polygonal domain with a piecewise smooth interface Γ. The conductivity, κ is piecewise constant so that $\kappa = \kappa_\pm$ on each $\Omega_\pm$.

The finite element (FE) formulation for (1.1) traces back to Babuska et al. [1–3]. They developed the partition of unity FE methods in which the finite elements are constructed by solving the interface problem locally. The local basis functions in these methods are able to capture very well the important features of the exact solution and they can be non-polynomials. Bramble and King derived a finite element method in which the smooth boundary and interface of the problem domain are approximated by polygonal domain and interface [4]. Later, the immersed finite element method (IFE) was introduced, where they allow the interface to cut through the element and the local basis functions constructed to satisfy the interface jump conditions of normal fluxes. IFE methods do not locally solve the interface problem and their basis functions are piecewise polynomials [9,10,14–17].

It is known that the finite volume method produces physically more relevant solutions for evolution equations than the usual finite element does. There have been studies in this direction for interface problems in the name of the immersed

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finite volume method [8,13]. The purpose of our paper is to introduce a P_1-nonconforming finite element induced by hybridization and a post processing to recover flux conserving numerical fluxes. By hybridization, we mean a construction of the linear system using flux continuity on the support of a local basis function. The major advantage of hybridization is that it produces flux preserving numerical schemes like a finite volume method, however it does not need a control volume generation. For details of hybridized methods we refer to [6,11,12]. As observed in [11,12], the P_1 and P_2 type hybridized methods yield symmetric linear systems for problems without an immersed interface. Especially, for a nonconforming P_1 method the hybridized method results in a symmetric nonconforming finite element system with a modified right hand side. A direct hybridization of immersed finite element method for interface problems yields a nonsymmetric linear system due to the interface elements. Non-symmetry of a linear system can cause difficulties in developing fast convergent iterative schemes.

In this paper we consider a modification of the hybridized method to obtain a symmetric stiffness system. The modification is needed only for elements with an immersed interface. The modification is composed of two procedures: (1) conversion of the nonsymmetric hybridized system into a symmetric nonconforming finite element system by using the Riesz representation, (2) a postprocessing to recover flux by an inverse Riesz representation so that it satisfies intercell flux continuity.

The paper is organized as follows. In Section 2, the function spaces, triangulation and its skeleton, and a hybridization approach are described. In Section 3, a conversion of a hybridized method into a typical nonconforming finite element method by using the Riesz representation is introduced. An analysis in the energy norm is provided in Section 4. In Section 5, we consider the rectangular elements. It is not difficult to see that the analysis in the previous section for triangular elements can be extended directly. In Section 6, we provide numerical results for simple elliptic interface problems by varying conductivity ratio. Numerical experiments are performed for both triangular and rectangular triangulations.

2. Hybridization

Let us first introduce triangulations and functional spaces. Let $\mathcal{T}_h$ be a shape regular, quasi-uniform triangular (or rectangular in Section 5) triangulation of Ω, where $\max_{K\in\mathcal{T}_h} diam(K) = h$. The skeleton K_h of a triangulation $\mathcal{T}_h$ is

$$K_h = \bigcup_{e\in\mathcal{E}_h} e,$$

where $\mathcal{E}_h$ is the set of edges. When the interface Γ trespasses a triangle T, it is called an *(immersed) interface triangle*. Otherwise, it is a noninterface triangle.

Let $H^m(D) = W_2^m(D)$ be the usual Sobolev space of order m with the norm $\|\cdot\|_{m,D}$. Here, $D \subset \mathbb{R}^2$ can be the whole domain Ω or a triangle T. The optimal function space for strong solutions of (1.1) is

$$H^1_{div}(\Omega) = \{u \in H^1(\Omega)\colon\ div\,(\kappa\nabla u) \in L_2(\Omega)\}.$$

For our numerical purpose we introduce the space $\widetilde{H}^2(D) \subset H^1_{div}(D)$ such that

$$\widetilde{H}^2(D) := \{u \in H^1(D)\colon\ \kappa\nabla u \in [H^1(D)]^2\},$$

equipped with the norm

$$\|u\|^2_{\widetilde{H}^2(D)} := \|u\|^2_{1,D} + \|\kappa\nabla u\|^2_{1,D}.$$

In the finite element analysis we require a regularity of solution

$$\|u\|^2_{H^2(\Omega_+\cup\Omega_-)} = \|u\|^2_{1,\Omega} + \|u\|^2_{2,\Omega_+} + \|u\|^2_{2,\Omega_-} < \infty$$

with the interface condition, $[[\partial_\nu^\kappa u]]_\Gamma = (\kappa_+\frac{\partial u}{\partial\nu_+} + \kappa_-\frac{\partial u}{\partial\nu_-})|_\Gamma = 0$ to have an optimal order of convergence. However, in our approach we require a stronger regularity $u \in \widetilde{H}_2(\Omega)$ for an optimal convergence analysis.

We denote the skeleton trace of $H^1(\Omega)$ by $H^{1/2}(K_h)$ and that of $H^1_0(\Omega)$ by $H^{1/2}_0(K_h)$. By the nature of nonconforming methods our analysis is based on the discrete Sobolev space $H^1(\mathcal{T}_h) = \prod_{T\in\mathcal{T}_h} H^1(T)$ with the norm and seminorm:

$$\|u\|^2_{1,h} := \sum_{T\in\mathcal{T}_h} \|u\|^2_{1,T}, \qquad |u|^2_{1,h} := \sum_{T\in\mathcal{T}_h} |u|^2_{1,T}.$$

The discrete inner product is given as

$$(\kappa\nabla u, \nabla v)_h = \sum_{T\in\mathcal{T}_h} (\kappa\nabla u, \nabla u)_T.$$

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