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Periodized radial basis functions, part I: Theory

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ABSTRACT

We extend the theory of periodized RBFs. We show that the imbricate series that define the Periodic Gaussian (PGA) and Sech (PSech) basis functions are Jacobian theta functions and elliptic functions "dn", respectively. The naive periodization fails for the Multiquadric and Inverse Multiquadric RBFs, but we are able to define periodic generalizations of these, too, by proving and exploiting a generalization of the Poisson Summation Theorem. Although applications of periodic RBFs are mostly left for another day, we do illustrate the flaws and potential by solving the Mathieu eigenproblem on both uniform and highly-adapted grids. The terms of a Fourier basis can be grouped into four classes, depending upon parity with respect to both the origin and $x = \pi/2$, and so, too, the Mathieu eigenfunctions. We show how to construct symmetrized periodic RBFs and illustrate these by solving the Mathieu problem using only the periodic RBFs of the same symmetry class as the targeted eigenfunctions. We also discuss the relationship between periodic RBFs and trigonometric polynomials with the aid of an explicit formula for the nonpolynomial part of the Periodic Inverse Quadratic (PIQ) basis functions. We prove that the rate of convergence for periodic RBFs is geometric, that is, the error can be bounded by $\exp(-N\mu)$ for some positive constant μ. Lastly, we prove a new theorem that gives the periodic RBF interpolation error in Fourier coefficient space. This is applied to the "spectral-plus" question. We find that periodic RBFs are indeed sometimes orders of magnitude more accurate than trigonometric interpolation even though it has long been known that RBFs (periodic or not) reduce to the corresponding classical spectral method as the RBF shape parameter goes to 0. However, periodic RBFs are "spectral-plus" only when the shape parameter α is adaptively tuned to the particular $f(x)$ being approximated and even then, only when $f(x)$ satisfies a symmetry condition.

1. Introduction

Radial basis functions (RBFs) have become a powerful tool for all sorts of applications in neural networks, solving partial differential equations in geometrically-complex domains, interpolation of scattered data on an irregular grid and so on as we shall review further below. However, little has been done with RBFs when the domain is spatially periodic. In this work, we extend the theory of periodic RBFs.

A nonperiodic function that decays as $|x| \to \infty$, including as a special case the RBF species function, $\phi(x)$, can always be "periodized" by making it the "pattern function" in an "imbricate series" [4,7] of the form

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Notation

$ce_{2n}(x; q)$	doubly symmetric Mathieu eigenfunctions
$ce_{2n+1}(x; q)$	Mathieu eigenfunctions symmetric about $x = 0$, antisymmetric about $x = \pi/2$
$se_{2n}(x; q)$	Mathieu eigenfunctions antisymmetric about $x = 0$ and $x = \pi/2$
$se_{2n+1}(x; q)$	Mathieu eigenfunctions antisymmetric about $x = 0$, symmetric about $x = \pi/2$
q	parameter of Mathieu equation
GA	Gaussian RBFs, $\phi(r) = \exp(-\epsilon^2 r^2)$
IMQ	Inverse Multiquadric RBFs $\phi(r) = \frac{1}{\sqrt{1+\epsilon^2 r^2}}$
IQ	Inverse Quadratic RBFs $\phi(r) = 1/(1 + \epsilon^2 r^2)$
MQ	Multiquadric RBFs $\phi(r) = \frac{1}{\sqrt{1+\epsilon^2 r^2}}$
N	number of interpolation points
PGA	Periodic Gaussian RBFs
PIMQ	Periodic Inverse Multiquadric RBFs
PIQ	Periodic Inverse Quadratic RBFs
PMQ	Periodic Multiquadric RBFs
Psech	Periodic Hyperbolic Secant (Sech) RBFs
Sech	Hyperbolic Secant (Sech) RBFs $\phi(r) = \text{sech}(\epsilon r)$
α	relative shape parameter
ϵ	absolute shape parameter (inverse width)
λ	eigenvalue of Mathieu function
$\phi(r)$	RBF species on an unbounded interval
$\theta(x)$	periodized RBF basis function

$$\theta(x) \equiv \sum_{m=-\infty}^{\infty} \phi(x - 2\pi m) \tag{1}$$

where we have normalized the period to 2π. The imbricate series is obviously periodic; the Poisson Summation Theorem, stated formally in Appendix A, shows that the Fourier coefficients of θ are the Fourier transform of the pattern function ϕ.

We shall explicitly discuss five species of periodic RBFs. Imbricating a Gaussian RBF gives a basis function which is a Jacobian theta function. Similar periodization of the Inverse Quadratic gives a function which is known in explicit form as the reciprocal of a linear trigonometric polynomial as well as a Fourier series whose coefficients are all powers of a parameter. Periodic Sech RBFs are Jacobian elliptic functions "dn".

To obtain generalizations of the popular Multiquadric and Inverse Multiquadric RBFs, we were forced to prove a new generalization of the Poisson Summation Theorem (Appendix B). Closed forms are not known, but the Periodic Multiquadric (PMQ) and Periodic Inverse Multiquadric (PIMQ) basis functions are given by explicit Fourier series and also a (modified) imbricate series in both cases.

The Mathieu eigenproblem is used to illustrate both the potential of adaptive grids and the value of symmetry-exploiting basis functions. We explicitly construct modified periodic RBFs that respect each of the four double-parity classes of the Mathieu eigenfunctions, thus reducing one large matrix eigenproblem (after discretization of the Mathieu differential equation) of size N into four small algebraic eigenproblems each of dimension $N/4$.

We also use the Mathieu problem to illustrate that RBFs do have the minor flaw that the spurious eigenvalues, an inevitable concomitant of discretized differential equation eigenproblems, are here intermixed with the "good" eigenvalues instead of being always large, as true of Fourier discretization. We also demonstrate the potential of adaptive grids with grid-tolerant RBFs by solving the eigenproblem for large parameter q using a grid that clusters interpolation points around the very narrow peaks of the eigenfunctions.

We begin with a brief review.

2. Background and review

2.1. RBFs

The approximation to a function using radial basis functions (RBFs), in any number of dimensions d, is

$$f(\vec{x}) \approx \sum_{j=1}^{N} a_j \phi(\|\vec{x} - \vec{c}_j\|), \quad \vec{x} \in R^d \tag{2}$$

for some "kernel" $\phi(r)$, always a *univariate* function even when the dimension $d > 1$, and some set of N points $\vec{c}_j$, which are called the "centers". The symbol $\| \|$ denotes the usual distance norm, $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots x_d^2}$. The coefficients a_j are usually found by interpolation at a set of points $\vec{x}_i$, which are usually chosen to coincide with the centers. The coefficients are computed by solving the matrix problem

$$\vec{\vec{V}} \vec{a} = \vec{f} \tag{3}$$

where matrix $\vec{\vec{V}}$ is the interpolation matrix, also called the "Vandermonde" or "Gram" matrix, with entries $\vec{\vec{V}}_{i,j} = \phi(\|\vec{x}_i - \vec{x}_j\|)$; $\vec{f}$ is the vector whose entries are $f(\vec{x}_i)$.