Applied Numerical Mathematics 86 (2014) 74–80

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A two-level higher order local projection stabilization on hexahedral meshes

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ARTICLE INFO

Article history:
Received 28 January 2014
Received in revised form 27 May 2014
Accepted 23 July 2014
Available online 7 August 2014

Keywords:
Local projection stabilization
Finite elements
Two-level approach

ABSTRACT

The two-level local projection stabilization with the pair $(Q_{r,h}, Q^{\text{disc}}_{r-1,2h})$, $r \geq 1$, of spaces of continuous, piecewise (mapped) polynomials of degree r on the mesh $\mathcal{T}_h$ in each variable and discontinuous, piecewise (mapped) polynomials of degree $r-1$ on the macro mesh $\mathcal{M}_h$ in each variable satisfy a local inf–sup condition leading to optimal error estimates. In this note, we show that even the pair of spaces $(Q_{r,h}, Q^{\text{disc}}_{r,2h})$, $r \geq 2$, with the enriched projection space $Q^{\text{disc}}_{r,2h}$ satisfies the local inf–sup condition and can be used in this framework. This gives a new, alternative proof of the inf–sup condition for the pair $(Q_{r,h}, Q^{\text{disc}}_{r-1,2h})$ in higher order cases $r \geq 2$.

1. Introduction

The local projection stabilization (LPS) is a popular method for solving scalar convection–diffusion–reaction equations, the Stokes problem and the Oseen problem [1–3,7–11,14,16,18]. There are different versions of the local projection stabilization on the market; here we will consider the two-level approach based on a standard finite element space Y_h on a mesh $\mathcal{T}_h$ and on projection spaces D_h living on a macro mesh $\mathcal{M}_h$. Hereby, the finer mesh is generated from the macro mesh by applying a certain refinement rule. For a convection–diffusion–reaction equation of type

$$-\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega,$$

the discrete problem of the LPS reads:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\varepsilon(\nabla u_h, \nabla v_h) + (b\cdot\nabla u_h + cu_h, v_h) + \sum_{M\in\mathcal{M}_h} \tau_M(\kappa_M\nabla u_h, \kappa_M\nabla v_h)_M = (f, v_h).$$

Here, $\tau_M \geq 0$ are user-chosen parameters, $V_h := Y_h \cap H^1_0(\Omega)$ denotes the ansatz space, $(\cdot,\cdot)_M$ is the inner product in $L^2(M)$ and its vector-valued version, respectively [4]. In case that $M = \Omega$ we omit the index M and write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_\Omega$. Let $\pi_h : L^2(\Omega) \to D_h$ be the L^2-projection into the discontinuous projection space D_h and $\kappa_h := id - \pi_h$ be the fluctuation operator, where its application to a vector has to be understood component-wise.

A fundamental property in the convergence analysis of the LPS is the L^2-orthogonality of the interpolation error in the ansatz space to the discontinuous projection space. In [13], it has been shown that a local inf–sup condition between ansatz and projection space is sufficient to construct modifications of standard interpolations which satisfy this additional orthogonality and lead to optimal error estimates in the convection-dominated case $\varepsilon \ll \|b\|_{0,\infty}$.

http://dx.doi.org/10.1016/j.apnum.2014.07.004
0168-9274/

Different refinement rules for generating the mesh $\mathcal{T}_h$ from the macro mesh $\mathcal{M}_h$ can be found in the literature. For example, in [6,13,17] each hexahedral macro cell $M \in \mathcal{M}_h$ is divided into 2^d hexahedral cells $K \in \mathcal{T}_h$ using the multi-linear reference mapping $F_M : \widehat{M} \to M$ from the reference hyper-cube $\widehat{M} = (-1,+1)^d$ onto M and their refined 2^d congruent hyper-cubes onto $K \subset M$. Then, as shown in [13, Lemma 3.2] the pairs $(Y_h, D_h) = (Q_{r,h}, Q^{\text{disc}}_{r-1,2h})$, $r \ge 1$, of spaces of continuous, piecewise (mapped) polynomials of degree r on $\mathcal{T}_h$ in each variable and discontinuous, piecewise (mapped) polynomials of degree $r-1$ on $\mathcal{M}_h$ in each variable satisfy the local inf–sup condition and can be used in the LPS framework. Let us define the local spaces

$$D_h(M) := \{q_h|_M : q_h \in D_h\}, \qquad Y_h(M) := \{v_h|_M : v_h \in Y_h\} \cap H^1_0(M).$$

Then, the local inf–sup condition between ansatz and projection space reads

$$\exists \beta > 0,\ \forall h > 0,\ \forall M \in \mathcal{M}_h \quad \inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(q_h, v_h)_M}{\|v_h\|_{0,M}\,\|q_h\|_{0,M}} \ge \beta \tag{1}$$

where $\|\cdot\|_{0,M}$ denotes the norm in $L^2(M)$. Roughly speaking (1) means that the bubble part of the ansatz space $Y_h(M)$ has to be rich enough compared to the local projection space $D_h(M)$, in particular, $\dim Y_h(M) \ge \dim D_h(M)$ is a necessary condition for (1). Although for the pairs $(Y_h, D_h) = (Q_{r,h}, Q^{\text{disc}}_{r,2h})$ with the larger projection spaces $Q^{\text{disc}}_{r,2h} \supset Q^{\text{disc}}_{r-1,2h}$ the necessary condition

$$\dim Y_h(M) = (2r-1)^d \ge (r+1)^d = \dim D_h(M) \quad r \ge 2,$$

is satisfied, the validity of the inf–sup condition (1) was unknown [13, Remark 3.4]. In this note, we give a positive answer to this open problem for sequences of uniformly refined meshes $\mathcal{M}_h$. Note that if the inf–sup condition is satisfied for the pair $(Q_{r,h}, Q^{\text{disc}}_{r,2h})$, $r \ge 2$, it is also satisfied for any pair $(Q_{r,h}, D_h)$ with $D_h \subset Q^{\text{disc}}_{r,2h}$ and $r \ge 2$. Thus, in particular, a new alternative proof of the inf–sup condition for the pair $(Q_{r,h}, Q^{\text{disc}}_{r-1,2h})$ in higher order cases $r \ge 2$ is given. Moreover, our approach offers the use of reduced two-level approaches in the spirit of [8,19].

2. The one-dimensional case

Let $\widehat{M} = (-1,+1)$ be the reference macro, $\widehat{K}_- = (-1,0)$, $\widehat{K}_+ = (0,1)$, and $F_M : \widehat{M} \to M$ the affine mapping of $\widehat{M}$ onto the macro cell $M \in \mathcal{M}_h$. The set of macro cells decompose the computational domain $\Omega \subset \mathbb{R}$. We define the spaces

$$\widehat{P}_{r,h} := \left\{\hat v \in H^1(\widehat{M}) : \hat v|_{\widehat{K}_-} \in P_r(\widehat{K}_-),\ \hat v|_{\widehat{K}_+} \in P_r(\widehat{K}_+)\right\}, \qquad \widehat{P}_{r,2h} := P_r(\widehat{M}),$$

where $\dim \widehat{P}_{r,h} = 2r+1$ and $\dim \widehat{P}_{r,2h} = r+1$. Consider the set of nodal functionals

$$N_i(\hat v) := \int_{-1}^{+1} \hat v(\xi) L_i(\xi)\, d\xi, \quad i = 0,1,\dots,r,$$

$$N_{r+1}(\hat v) := \hat v(-1), \qquad N_{r+2}(\hat v) := \hat v(+1),$$

where L_i, $i = 0,1,\dots$, denote the Legendre polynomials of degree i on $(-1,+1)$ normalized such that $L_i(1) = 1$. The first $r+1$ nodal functionals guarantee that a local interpolation $\widehat{J} : H^1(\widehat{M}) \to \widehat{P}_{r,h}$, defined by $N_i(\hat v - \widehat{J}\hat v) = 0$ for $i = 0,\dots,r+2$, satisfies the orthogonality property

$$(\hat v - \widehat{J}\hat v, q)_{\widehat{M}} = 0 \quad \text{for all } q \in P_r(\widehat{M}),\ \hat v \in H^1(\widehat{M}).$$

The last two nodal functionals secure that the interpolation can be extended to a global continuous interpolation $j_h : H^1_0(\Omega) \to V_h$ with the desired properties. However, because of $\dim \widehat{P}_{r,2h} < r+3$ we cannot hope to find an interpolation $\widehat{J} : H^1(\widehat{M}) \to \widehat{P}_{r,2h}$ into the coarse space $\widehat{P}_{r,2h}$ satisfying all $r+3$ conditions. We will show that a suitable enrichment of $\widehat{P}_{r,2h}$ by just two additional functions from $\widehat{P}_{r,h}$ is enough to meet these requirements. Let us consider the following functions

$$\hat\varphi_r(x) := \begin{cases} \Lambda_r(x) + \Lambda_{r-1}(x) & x \in [-1,0], \\ \Lambda_r(-x) + \Lambda_{r-1}(-x) & x \in [0,+1], \end{cases}$$

$$\hat\psi_r(x) := \begin{cases} \Lambda_r(x) - \Lambda_{r-2}(x) & x \in [-1,0], \\ -(\Lambda_r(-x) - \Lambda_{r-2}(-x)) & x \in [0,+1], \end{cases}$$

where Λ_r denotes the Legendre polynomial of degree r on $(-1,0)$ given by

$$\Lambda_r(x) = L_r(2x+1), \quad x \in (-1,0).$$

Furthermore, we introduce the linear mapping $\Phi : \widehat{P}_{r,h} \to \mathbb{R}^{r+3}$ given by

$$\Phi(\hat v) = \left(N_0(\hat v), \dots, N_{r+2}(\hat v)\right).$$