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Inflow-implicit/outflow-explicit finite volume methods for solving advection equations

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ABSTRACT

We introduce a new class of methods for solving non-stationary advection equations. The new methods are based on finite volume space discretizations and a semi-implicit discretization in time. Its basic idea is that outflow from a cell is treated explicitly while inflow is treated implicitly. This is natural, since we know what is outflowing from a cell at the old time step but we leave the method to resolve a system of equations determined by the inflows to a cell to obtain the solution values at the new time step. The matrix of the system in our inflow-implicit/outflow-explicit (IIOE) method is determined by the inflow fluxes which results in an M-matrix yielding favorable stability properties for the scheme. Since the explicit (outflow) part is not always dominated by the implicit (inflow) part and thus some oscillations can occur, we build a stabilization based on the upstream weighted averages with coefficients determined by the flux-corrected transport approach [2,19] yielding high resolution versions, S^1IIOE and S^2IIOE, of the basic scheme. We prove that our new method is exact for any choice of a discrete time step on uniform rectangular grids in the case of constant velocity transport of quadratic functions in any dimension. We also show its formal second order accuracy in space and time for 1D advection problems with variable velocity. Although designed for non-divergence free velocity fields, we show that the basic IIOE scheme is locally mass conservative in case of divergence free velocity. Finally, we show L^2-stability for divergence free velocity in 1D on periodic domains independent of the choice of the time step, and L^∞-stability for the stabilized high resolution variant of the scheme. Numerical comparisons with the purely explicit schemes like the fully explicit up-wind and the Lax–Wendroff schemes were discussed in [13] and [14] where the basic IIOE was originally introduced. There it has been shown that the new scheme has good properties with respect to a balance of precision and CPU time related to a possible choice of larger time steps in our scheme. In this contribution we compare the new scheme and its stabilized variants with widely used fully implicit up-wind method. In this comparison our new schemes show better behavior with respect to stability and precision of computations for time steps several times exceeding the CFL stability condition. Our new stabilized methods are L^∞ stable, second order accurate for any smooth solution and with accuracy of order 2/3 for solutions with moving discontinuities. This is opposite to implicit up-wind schemes which have accuracy order 1/2 only. All these properties hold for any choice of time step thus making our new method attractive for practical applications.

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1. Introduction

In this article we present the inflow-implicit/outflow-explicit (IIOE) method, and its stabilized (S^1IIOE, S^2IIOE) high-resolution variants, for solving general time dependent variable velocity advection equations of the form

$$u_t + \mathbf{v} \cdot \nabla u = 0 \tag{1}$$

where $u \in \mathbb{R}^d \times [0, T]$ is the unknown function and $\mathbf{v}$ is a vector field which may vary in space, e.g. $\mathbf{v} = \mathbf{v}(x, u, \nabla u)$. Variable velocity vector fields arise in many applications, e.g. in transport equations with non-divergence free velocities or nonlinear conservation laws [11], in the Eulerian level set methods for evolving fronts [18], in a tangentially stabilized Lagrangean methods for evolving interfaces [1,16] or in other applications like an image segmentation by active contours in a form of generalized subjective surface method [3,4,9,15,17,20]. In such case of image segmentation, the spatially varying vector field $\mathbf{v}(x)$ depends on the gradient of the image intensity function. For motion of level sets in normal direction with speed $F(x)$ we have $\mathbf{v} = F(x)\frac{\nabla u}{|\nabla u|}$ [12,18]. In this case the basic IIOE method coincides with the semi-implicit forward–backward diffusion approach recently presented by the authors in [12]. In the context of level set equations the originality of the approach in [12] consists in rewriting the level set equation for motion in normal direction in terms of an equation containing the forward and backward diffusion. Then naturally, the forward diffusion dominated parts of the model are treated implicitly, while the backward diffusion dominated parts are treated explicitly. The resulting scheme is a semi-implicit second order numerical scheme that allows large time steps. Hence, our new IIOE method presented in [13,14] and in this paper can be seen as a generalization of the approach from [12] to arbitrary variable velocity advection equations.

The basic idea of our new IIOE method is that outflow from a cell is treated explicitly while inflow is treated implicitly. Such an approach is natural, since we know what is flowing out from a cell at an old time step $n-1$ but we leave the method to resolve a system of equations determined by the inflows to obtain a new value in the cell at time step n. Since the matrix of the system is determined by the inflow fluxes, it is an M-matrix and thus it has favorable solvability and stability properties. It is worth to note that a similar idea to construct the M-matrix in the implicit part has also been introduced by Kuzmin and his co-authors by using a purely algebraic approach in the context of solving advection equations by the Galerkin finite element method combined usually with the Crank–Nicolson time stepping, see [10] for the latest state-of-the-art and related references.

Since the explicit (outflow) part is not always dominated by the implicit (inflow) part, some oscillations can occur in the basic IIOE scheme. One way is to leave them propagate and perform some postprocessing of the numerical solution, or another way is to incorporate a stabilization mechanism into the scheme itself. As it was shown in [12], a special local averaging was sufficient to stabilize the forward–backward diffusion approach in order to get stable second order solution in case of smooth level set interface motion in normal direction, but in general, such local averaging does not guarantee fulfilling sharply the discrete minimum–maximum principle. In this paper we build a new stabilization of the basic IIOE scheme based on the so-called flux-corrected transport approach [2,19] yielding L^∞ stable high resolution variant of the scheme.

We also present theoretical results for our new scheme, namely, its exactness for any choice of time step on uniform rectangular grids in the case of constant velocity transport of quadratic functions in any dimension and its formal second order accuracy in space and time for 1D advection problems with variable velocity. Although designed for non-divergence free velocity fields, we show that the basic IIOE scheme is locally mass conservative in case of divergence free velocity. Finally, we show L^2-stability for divergence free velocity in 1D on periodic domains independent of the choice of the time step, and L^∞-stability for the stabilized high resolution variant of the scheme.

Numerical comparisons with the fully explicit schemes like the fully explicit up-wind and the (limited) Lax–Wendroff method were discussed in [13] and [14]. There, the positive properties of the new scheme have been shown with respect to a balance of precision and CPU time. Thus, in this paper we concentrate mainly on a comparison of the new IIOE scheme and its stabilized variants with the well-known and widely used fully implicit up-wind method for solving advection equations. We show superior behavior of our new schemes with respect to stability and precision of computations for time steps largely exceeding CFL stability condition. Our new schemes are L^∞ stable, second order accurate for smooth solutions and with accuracy of order 2/3 for solutions with moving discontinuities, opposite to the implicit up-wind schemes which have accuracy order only 1/2 in the discontinuous case. Moreover, all these properties hold for any choice of time step and thus make our new methods attractive from the point of view of practical applications where no concern on CFL restrictions is preferable.

The rest of the article is organized as follows. In Section 2 we introduce the general formulation of the basic and stabilized IIOE schemes on unstructured grids in several space dimensions. In Section 3, for clarity and also due to the reasons of our theoretical study, we write a 1D version of the IIOE scheme and also its higher dimensional form in the case of advective motion of level sets in normal direction in order to clearly see its relation to the forward–backward diffusion approach from [12]. In Section 4 we present theoretical results for our new schemes and in Section 5 we present several representative 1D and higher dimensional numerical experiments demonstrating interesting properties of the method.