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Multiscale approach for stochastic elliptic equations in heterogeneous media ☆

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ABSTRACT

This paper presents a multiscale analysis for stochastic elliptic equations in heterogeneous media. The main contributions are threefold: derive the convergence rate of the first-order asymptotic solution based on the periodic approximation method; develop a new technique for dealing with a large stochastic fluctuation; and present a novel multiscale asymptotic method. A multiscale finite element method is developed, and numerical results for solving stochastic elliptic equations with rapidly oscillating coefficients are reported.

1. Introduction

In this paper, we consider the stochastic elliptic equation with rapidly oscillating coefficients given by

$$\begin{cases} \mathcal{L}_\varepsilon u^\varepsilon(x,\omega) \equiv -\dfrac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x,\omega)\dfrac{\partial u^\varepsilon(x,\omega)}{\partial x_j}\right) = f(x), & (x,\omega)\in\Omega\times\Theta, \\ u^\varepsilon(x,\omega) = g(x), & (x,\omega)\in\partial\Omega\times\Theta, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bounded convex domain, and $(\Theta, \mathcal{F}, \mu)$ is a standard probability space.

We make the following assumptions:

(A₁) $(a_{ij}^\varepsilon(x,\omega))$ is a symmetric matrix and $\mu(\omega\in\Theta : a_{ij}^\varepsilon(x,\omega)\in[\gamma_0,\gamma_1], \forall x\in\Omega) = 1$, γ_0, γ_1 are non-random positive constants.

(A₂) $a_{ij}^\varepsilon \in L^2(\Theta, d\mu; L^\infty(\Omega))$, where $L^2(\Theta, d\mu; L^\infty(\Omega))$ is the Lebesgue space with respect to a product measure μ, see [30]. $f\in L^2(\Omega)$, $g\in H^{\frac{1}{2}}(\partial\Omega)$.

The model problem given in (1) has a wide range of applications, such as in heat and mass transfer of composite materials or porous media, and in fluid mechanics on heterogeneous media [31,33]. These problems typically involve materials

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with a large number of heterogeneities (inclusions or holes). In such cases, a direct numerical computation becomes extremely difficult, because it would require a very fine mesh to ensure the accuracy. The crucial idea of the homogenization method is to give the overall behavior of the composite by incorporating the fluctuations due to heterogeneities. The most general theory of the homogenization without any special assumptions on the problem geometry (such as the periodicity or randomness) is the H-convergence, see [27]. The H-convergence method applies to periodic homogenization and gives an explicit formula for this limit. Recently, Svanstedt [32] used the H-convergence to construct explicit formulas for elliptic and parabolic problems in random multiscale structures. Kozlov [22] (see also [29]) showed that, for almost all $\omega \in \Theta$ a matrix-valued, statistically homogeneous ergodic random field $A^{\varepsilon}(x,\omega)$ admits homogenization as $\varepsilon \to 0$, and the homogenization matrix $\widehat{\mathcal{A}}$ is independent of ω. The related problems were also studied in [19,31,33,35]. However, to our knowledge, only a few numerical studies have been conducted for the stochastic homogenization. Jardak and Ghanem [18] presented a formulation and numerical analysis of stochastic homogenization and the main feature is the characterization of the homogenized solution as a stochastic process, which is identified with its projection on a basis in a suitable Hilbert space. E and Engquist [10] proposed the overall framework of the heterogeneous multi-scale method (HMM). Furthermore, this method has been extended to random heterogeneities, see [11,12,24,25]. Li and Cui [23] employed a direct Monte Carlo method to calculate the effective elastic constants of the composite materials with a discrete random field. Luo [26] developed an upscaled Wiener chaos expansion (WCE) method to solve stochastic elliptic equations with rapidly oscillating coefficients. They further applied this method to the uncertainty quantification in a subsurface modeling. Kamiński and Kleiber [20] used the second order perturbation and the stochastic second central moment to solve the homogenization problem of two-phase elastic composites and also presented a mathematical formulation and numerical analysis for a homogenization of random elastic composites with stochastic interface defects, see [21]. It should be noted that numerous studies on numerical solutions for SPDEs without a small parameter ε have been reported, and it is impossible to mention all contributions here. We refer the interested reader to references listed in [1–3,14–16,30,34].

We recall that a homogenization describes the asymptotic behavior of the solution to the problem as $\varepsilon \to 0$. However, in many engineering applications, while ε is small, it does not approach zero. Numerous numerical results (e.g. [8,9]) have shown that the numerical accuracy of the standard homogenization method may not be satisfactory if ε is not sufficiently small. It is our motivation to introduce the multiscale asymptotic methods for solving SPDEs with rapidly oscillating coefficients.

The new contributions addressed in this paper are as follows. Using Bourgeat's idea on the periodic approximation of a random media, we derive the convergence rate of the first-order asymptotic solution for problem (1) based on the homogenization result of [6]. In principle, the validity of a perturbation method is restricted to cases where the random elements exhibit only small fluctuations about their mean values. In this paper, we present a new technique for dealing with large fluctuations, and thanks to the theoretical results of [13], we obtain the convergence rates for the Karhunen–Loève expansion and the modified Neumann expansion. If the right-hand-side of the equation contains a small parameter ε and random variables, then the classical multiscale asymptotic methods cannot be employed [4,9,19]. In this study, we present a novel multiscale asymptotic method and derive the convergence results. The crucial step is the determination of the corrector terms. Finally, a multiscale finite element method is developed and numerical simulations are reported.

Denote uniformly by C the positive non-random constant independent of ε. For convenience, we use the Einstein summation convention for the repeated indices.

2. Periodic approximation of random media

We first introduce Bourgeat's result of a periodic approximation for the random media, see [6]. To this end, we need to make the following assumption:

(A_3) $A^{\varepsilon}(x,\omega) = (a^{\varepsilon}_{ij}(x,\omega))$ is a matrix-valued, statistically homogeneous ergodic random field, see [19,22,29].

We now recall some definitions and notation. Let $(\Theta, \mathcal{F}, \mu)$ be a standard probability space, and assume that an n-dimensional dynamical system T_z, $z \in R^n$, is given on Θ, i.e. a family of invertible maps $T_z : \Theta \to \Theta$, $z \in R^n$, such that

(i) $T_{y+z} = T_y T_z$, $T_0 = I$, where I is an identify operator;

(ii) T_z preserves the measure μ, i.e. $\mu(T_z^{-1}(\mathcal{A})) = \mu(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{F}$ and any $z \in R^n$;

(iii) T_z is a measurable mapping from $R^n \times \Theta$ to Θ, where $R^n \times \Theta$ is equipped with the product σ-algebra $\mathcal{B} \times \mathcal{F}$ and $\mathcal{B}$ is the Borel σ-algebra in R^n.

For such a dynamical system, a large class of statistically homogeneous random fields can be introduced as follows. For an arbitrary random variable $f = f(\omega)$, we define $f(z,\omega) \equiv f(T_z\omega)$. It is easy to verify that $f(z,\omega)$ is a statistically homogeneous random field. We suppose that the coefficients of the random operators are defined in terms of a dynamical system T_z.

The definition of a uniform mixing condition [6] can be described as follows. Given a statistically homogeneous random field $\zeta(z,\omega)$ in R^n, we denote $\mathcal{F}_A$ the σ-algebra $\sigma\{\zeta(z), z \in A\}$. The function

$$\alpha(s) = \sup_{A,B \subset R^n, \operatorname{dist}(A,B) \ge s} \ \sup_{\mathcal{A} \in \mathcal{F}_A,\, \mathcal{B} \in \mathcal{F}_B} \left|\mu(\mathcal{A} \cap \mathcal{B}) - \mu(\mathcal{A})\mu(\mathcal{B})\right|$$