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A modified Nyström–Clenshaw–Curtis quadrature for integral equations with piecewise smooth kernels ☆

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ABSTRACT

The Nyström–Clenshaw–Curtis (NCC) quadrature is a highly accurate quadrature which is suitable for integral equations with semi-smooth kernels. In this paper, we first introduce the NCC quadrature and point out that the NCC quadrature is not suitable for certain integral equation with well-behaved kernel functions such as $e^{-|t-s|}$. We then modify the NCC quadrature to obtain a new quadrature which is suitable for integral equations with piecewise smooth kernel functions. Applications of the modified NCC quadrature to Wiener–Hopf equations and a nonlinear integral equation are presented.

1. Introduction

Integral equations occur in various areas of applied mathematics, physics, and engineering. There are rich results in numerical methods for the solution of Fredholm integral equation of the second kind; see for instance [1,2,5,6,8–11,16–19, 25] and references cited therein. In this paper, we consider Fredholm integral equation of the second kind

$$x(t) + \int_a^b k(t,s)x(s)ds = y(t), \quad a \le t \le b, \tag{1}$$

where $k(t,s)$ has a discontinuity either by itself or in its partial derivatives along $t = s$ and $y(t)$ has continuous qth derivative with q being a positive integer. Expansion method is a good choice for handling the discontinuity in the kernel function, see [10,19]. However, the computational cost for obtaining the discretized linear system is quite large.

In 2003, Kang, Koltracht, and Rawitscher proposed a highly accurate numerical quadrature called Nyström–Clenshaw–Curtis (NCC) quadrature and applied it to Eq. (1) for the case where $k(t,s)$ is semi-smooth [18]. That is, there exists $p > 1$ such that

$$k(t,s) = \begin{cases} k_1(t,s), & a \le s \le t \le b, \\ k_2(t,s), & a \le t \le s \le b, \end{cases}$$

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with $k_i(t,s) \in C^p([a,b] \times [a,b])$, $i = 1, 2$. Such kind of kernel functions are said to be p-semi-smooth. We will introduce the NCC quadrature and discuss some properties of the quadrature in Section 2.

The aim of this paper is to modify the NCC quadrature such that the modified quadrature is suitable for piecewise smooth kernels. More precisely, our modified NCC quadrature is suitable for kernel functions belonging to

$$E^p\big([a,b] \times [a,b]\big) \triangleq \big\{k(t,s) \mid k_1(t,s) \in C^p\big(\big\{(t,s) \mid a \le s \le t \le b\big\}\big), k_2(t,s) \in C^p\big(\big\{(t,s) \mid a \le t \le s \le b\big\}\big)\big\}. \tag{2}$$

We note that any p-semi-smooth function belongs to $E^p([a,b] \times [a,b])$. However, there are functions in $E^p([a,b] \times [a,b])$ which are not p-semi-smooth, e.g.,

$$k(t,s) = 1/\big(1 + |t-s|^3\big), \quad 0 \le t, s \le 2.$$

The key points for modifying the NCC quadrature are: for fixed t, we extend $k_1(t,s)$ and $k_2(t,s)$ respectively such that the extended functions $\tilde{k}_1(t,s)$ and $\tilde{k}_2(t,s)$ (as functions of s) are smooth in the interval $[a,b]$, and then approximate $\tilde{k}_i(t,s)x(s)$ by polynomials of s.

The outline of the paper is as follows. In Section 2, we introduce the NCC quadrature and discuss some properties of the quadrature. In Section 3, we present a modified NCC quadrature. In Sections 4 and 5, we apply the modified NCC quadrature to Wiener–Hopf equations and a nonlinear integral equation, respectively. Finally, concluding remarks are given in Section 6.

2. The Nyström–Clenshaw–Curtis quadrature

In this section, we introduce the NCC quadrature in a different way.

Assume $k(t,s)$ is a p-semi-smooth kernel and $[a,b] = [-1,1]$. Rewrite (1) as

$$x(t) + \int_{-1}^{t} k_1(t,s)x(s)ds + \int_{t}^{1} k_2(t,s)x(s)ds = y(t), \quad -1 \le t \le 1. \tag{3}$$

Since $k_1(t,s)$ and $k_2(t,s)$ are smooth in $[-1,1] \times [-1,1]$, $k_1(t,s)x(s)$ and $k_2(t,s)x(s)$ are also smooth $[-1,1] \times [-1,1]$. The key point of the NCC quadrature is to approximate the integrands in (3) (as functions of s for fixed t) by polynomials in the whole interval $[-1,1]$.

Let $T_j(s) = \cos(j \arccos(s))$, $j = 0, 1, 2, \ldots$, be the sequence of Chebyshev polynomials and let

$$\tau_l = \cos\frac{(2l-1)\pi}{2n}, \quad l = 1, \ldots, n, \tag{4}$$

be the roots of $T_n(s) = \cos(n \arccos(s))$. Define

$$F_l(t) = \int_{-1}^{t} k_1(\tau_l, s)x(s)ds, \quad -1 \le t \le 1.$$

Interpolating $f_l(s) \triangleq k_1(\tau_l, s)x(s)$ at the Chebyshev nodes $\tau_1, \tau_2, \ldots, \tau_n$, we get an approximation of $f_l(s)$:

$$f_l(s) \approx \tilde{f}_l(s) = \sum_{j=1}^{n} \alpha_{l,j} T_{j-1}(s), \quad -1 \le s \le 1,$$

where

$$[\alpha_{l,1}, \cdots, \alpha_{l,n}]^T = C^{-1}\big[k_1(\tau_l, \tau_1)x(\tau_1), \cdots, k_1(\tau_l, \tau_n)x(\tau_n)\big]^T \tag{5}$$

with C being given by

$$C = \big[T_{j-1}(\tau_i)\big]_{i,j=1}^{n} = \left[\cos\frac{(2i-1)(j-1)\pi}{2n}\right]_{i,j=1}^{n}.$$

It follows that

$$F_l(t) \approx \int_{-1}^{t} \tilde{f}_l(s)ds = \int_{-1}^{t} \sum_{j=1}^{n} \alpha_{l,j} T_{j-1}(s)ds = \sum_{j=1}^{n} \alpha_{l,j} \int_{-1}^{t} T_{j-1}(s)ds. \tag{6}$$