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A posteriori error estimates for a discontinuous Galerkin method applied to one-dimensional nonlinear scalar conservation laws

Mahboub Baccouch

Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182, United States

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ABSTRACT

In this paper, new *a posteriori* error estimates for a discontinuous Galerkin (DG) formulation applied to nonlinear scalar conservation laws in one space dimension are presented and analyzed. These error estimates are computationally simple and are obtained by solving a local problem with no boundary condition on each element of the mesh. We first show that the leading error term on each element for the solution is proportional to a $(p+1)$-degree Radau polynomial, when p-degree piecewise polynomials with $p \geq 1$ are used. This result allows us to prove that, for smooth solutions, these error estimates at a fixed time converge to the true spatial errors in the L^2-norm under mesh refinement. The order of convergence is proved to be $p + 5/4$. Finally, we prove that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our computational results indicate that the observed numerical convergence rates are higher than the theoretical rates.

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1. Introduction

The discontinuous Galerkin (DG) method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [31]. Lesaint and Raviart [29] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary differential equations [7,18,28,29], hyperbolic [14–17,23,22,26,27] and diffusion and convection–diffusion [12,13,32] partial differential equations. Consult [24] and the references cited therein for a detailed discussion of the history of the DG method and a list of important citations on the DG method and its applications.

In recent years, the study of superconvergence and *a posteriori* error estimates of DG methods has been an active research field in numerical analysis. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Superconvergence properties for DG methods have been studied in [7,6,25,29] for ordinary differential equations, [3,9,7,5] for hyperbolic problems and [1,2,4,5,11,19–21] for diffusion and convection–diffusion problems.

A posteriori error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree

E-mail address: mbaccouch@unomaha.edu.

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is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [8].

The first superconvergence result for standard DG solutions of ordinary differential equations appeared in Adjerid et al. [7]. They proved that the p-degree DG solution of $u' - au = 0$ is $\mathcal{O}(h^{p+2})$ superconvergent at the roots of $(p+1)$-degree right Radau polynomial. Numerical computations indicate that these superconvergence results extend to DG solutions of transient convection problems.

Cheng and Shu [21] studied the superconvergence property for the DG and the local DG (LDG) methods for solving one-dimensional time-dependent linear conservation laws and convection–diffusion equations. They proved superconvergence towards a particular projection of the exact solution when the upwind flux is used for conservation laws and when the alternating flux is used for convection–diffusion equations. The order of superconvergence for both cases is proved to be $p + 3/2$ when p-degree piecewise polynomials with $p \geq 1$ are used. Later, Adjerid and Baccouch [3,10] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid et al. [7]. They proved that, for smooth solutions, these *a posteriori* error estimates at a fixed time t converge to the true spatial errors in the L^2-norm under mesh refinement.

More recently, Meng et al. [30] analyzed the superconvergence property of the DG method applied to one-dimensional time-dependent nonlinear scalar conservation laws. They proved that the error between the DG solution and a particular projection of the exact solution achieves $(p + 3/2)$-th order superconvergence when p-degree piecewise polynomials with $p \geq 1$ and upwind fluxes are used. The results in the present paper depend heavily on results from this reference.

The objectives of this work are: (i) to construct simple and efficient procedures to compute *a posteriori* error estimates of nonlinear conservation laws in one space dimension and (ii) to analyze the global convergence of these *a posteriori* error estimates, extending the results in [3] for linear problems. To the best knowledge of the author, this is the first convergence proof of an *a posteriori* error estimates for the one-dimensional nonlinear scalar conservation laws. We first use the superconvergence result of Meng et al. [30] to show that the true error on each element of the mesh can be divided into a significant and a less significant part. The significant part of the discretization error for the DG solution is proportional to $(p + 1)$-degree Radau polynomial and the L^2-norm of the less significant part is $\mathcal{O}(h^{p+3/2})$. Superconvergence results are used to construct asymptotically correct *a posteriori* error estimates by solving a local problem on each element of the mesh. We further show that the DG discretization error estimates converge to the true spatial errors under mesh refinement at $\mathcal{O}(h^{p+5/4})$ rate. Finally, we prove that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our computational results indicate that the observed numerical convergence rates are higher than the theoretical rates.

Our proofs are valid for any regular meshes and using piecewise polynomials of degree $p \geq 1$, provided that the flux function $|f'(u)|$ is lower bounded uniformly by a positive constant. The generalization to nonlinear equations with general flux functions is discussed at the end of Section 4. We included numerical examples to show that our results are valid for nonlinear conservation laws with general flux functions. The proof of the general case involves several technical difficulties and will be investigated in the future. For general flux functions, we expect that similar superconvergence results of Meng et al. [30] will be needed. The theoretical analysis of superconvergence in higher dimensions is a subject of ongoing research.

In our analysis time integration is assumed to be exact and thus we are only estimating the spatial error of the semi-discrete DG method. Thus, we present superconvergence and error estimates for the semi-discrete problem after discretizing in space only. The analysis of the fully discrete methods for nonlinear conservation laws will be reported in a forthcoming paper.

This paper is organized as follows: In Section 2 we present the semi-discrete DG method for solving the one-dimensional nonlinear conservation laws and we introduce some notations and definitions. In Section 3, we present the DG error analysis and prove our main superconvergence results. We also present few preliminary results, which will be needed in our *a posteriori* error analysis. In Section 4, we present *a posteriori* error estimation procedures and prove that these error estimates converge to the true errors under mesh refinement in L^2-norm. In Section 5, we present several numerical examples to validate our results. We conclude and discuss our results in Section 6.

2. The semi-discrete DG scheme

We consider the following nonlinear conservation laws in one space variable

$$u_t + \big(f(u)\big)_x = g(x,t), \quad x \in [-1,1],\ t \in [0,T], \tag{2.1a}$$

subject to the initial and periodic boundary conditions

$$u(x,0) = u_0(x), \quad x \in [-1,1], \tag{2.1b}$$

$$u(-1,t) = u(1,t), \quad t \in [0,T]. \tag{2.1c}$$

The assumption of periodic boundary conditions is not essential. Our results hold for Dirichlet boundary condition of the form $u(-1,t) = h(t)$.