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Solution of double nonlinear problems in porous media by a combined finite volume–finite element algorithm

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ABSTRACT

The combined finite volume–finite element scheme for a double nonlinear parabolic convection-dominated diffusion equation which models the variably saturated flow and contaminant transport problems in porous media is extended. Whereas the convection is approximated by a finite volume method (Multi-Point Flux Approximation), the diffusion is approximated by a finite element method. The scheme is fully implicit and involves a relaxation-regularized algorithm. Due to monotonicity and conservation properties of the approximated scheme and in view of the compactness theorem we show the convergence of the numerical scheme to the weak solution. Our scheme is applied for computing two dimensional examples with different degrees of complexity. The numerical results demonstrate that the proposed scheme gives good performance in convergence and accuracy.

1. Introduction

This paper is devoted to the study of a combined finite volume–finite element approximation scheme for a double nonlinear convection diffusion problem. Specifically, the model is described by:

$$\partial_t b(u) + \mathrm{div}\big(\mathbf{v}(x,t)u - \mathbf{D}(x,t)\beta(u)\nabla u\big) = f(x,t,u) \quad \text{on } Q_T, \tag{1.1}$$

where $Q_T \equiv \Omega \times I$, $I = (0, T]$, $T < \infty$, $\Omega \subset R^{\mathbf{d}}$, $\mathbf{d} = 2$, or $\mathbf{d} = 3$ is a bounded domain with Lipschitz continuous boundary. The above problem is provided with the following homogeneous Dirichlet boundary and initial conditions:

$$u(x,t)|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega \times I, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

The functions $b(u)$ and $\beta(u)$ are nonlinear and are assumed to be continuous and monotonic increasing in our problem (1.1). Further assumptions will be shown later.

The problem (1.1) arises in groundwater aquifers and oil reservoir applications that often requires multiple phase flow models [3,24], flows with phase change [21] and turbulent gas flowing in pipelines [8]. A special case of the model (1.1) is the water flow model in variably saturated rigid porous media which is given by Richards' equation:

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$$\partial_t\theta(h) = \nabla\cdot\big(K(\theta)\nabla h\big) + \partial_\theta K(\theta)\partial_z h - S(z,t) \quad \text{in } (\Omega\times I), \tag{1.4}$$

where h is the pressure head, θ is the volumetric water content, K is the hydraulic conductivity, $S(z,t)$ represents a source term and z is the soil depth. While Eq. (1.4) seems to be parabolic, it is considered hyperbolic when the gravitational term, the convection part of (1.4), is relatively important, see e.g. [29]. The terms $\theta(h)$ and $K(\theta)$ are nonlinear functions and can be described by the well-known parametric functions of van Genuchten [30], Brooks and Corey [28] and Storm and Fujita [4].

The above problem, (1.1), also captures the fundamental features of contaminant transport model in porous media with equilibrium adsorption:

$$\partial_t\big(\theta u + \rho\psi(u)\big) + \operatorname{div}(\mathbf{v}u - \mathbf{D}\nabla u) = 0 \quad \text{in } (\Omega\times I), \tag{1.5}$$

where u is the dissolved concentration, $\mathbf{v}$ is the (Darcy) velocity field of water, θ is the porosity of the medium, $\mathbf{D}$ is the diffusion–dispersion coefficient, ρ is the bulk density, $\mathbf{v}$ is the Darcy velocity of the fluid mixture and ψ is the solute sorption isotherm.

In order to solve numerically the model (1.1), there are two main issues that should be taken into account. Firstly, the fact that Eq. (1.1) is nonlinear (or degenerate) implies that the solution lacks regularity. Secondly, convection dominant diffusion demands the shock capturing profile method. Consequently, the numerical approximation of such solutions needs an efficient scheme to handle these difficulties. Recently many authors have discussed the above issues for the problem (1.1), for instance: an implicit finite volume method [11]; a regularized mixed finite element method [26], see also [2]; a relaxation iterative finite element method; [15] and a more recent work [18], where another variant of relaxation scheme is considered.

It is well known that the numerical modeling of the above problems for partially saturated porous media which have discontinuous profiles using finite element (FE) schemes is challenging. For example, this approach may lead to numerical diffusion and unphysical oscillations and thus to inaccurate water flow and the contaminant fate prediction. To address this issue, many attempts have been made to develop powerful conservative numerical schemes for convection dominant problems that minimize the numerical diffusion to capture the mathematical properties of physical systems.

In this regard, two approaches are relevant: a modified method of characteristics with adjusted advection scheme MMOCAA proposed by Douglas et al. [9] and developed by Mahmood [17,18] (we called it CMMOC) and a combined finite volume–finite element scheme (FV–FE) proposed by [13] (see also [12,14]) and developed by Mahmood et al. [19] for high order Godunov upwind scheme by approximating the advection term using monotonic MUSCL cell vertex second order finite volume method. The approximation procedure for these approaches, FV–FE methods and CMMOC, provides an efficient discretization of the transport and the diffusion terms to get conservative shock capturing schemes with a significantly reduced numerical diffusion and the ability to capture the flow features. Moreover, the methods can also be reformulated in the framework of mixed finite element discretization [12,7]. In addition finite volume–finite element method can be applied to the non-matching meshes which play a strong role in application e.g. in fractured media. A higher order of accuracy can be achieved for FV–FE, third order, through cell based polynomial reconstruction, see e.g. [5].

In the present paper we develop and analyze an efficient scheme which is based on the iterative relaxation scheme (an efficient approximation to the nonlinearity (in the parabolic term) that arises in these models), a combined finite volume–finite element scheme and a regularization to the nonlinear diffusion coefficient term, to solve the model (1.1). The convection term is approximated by using cell vertex finite volume method with upwind scheme whereas the diffusion is approximated by the nonconformal finite element method. A dual mesh scheme for finite volume is based on a mesh for finite element method. This scheme combines two properties: the conservation of the finite volume, that has been widely used for solving hyperbolic systems in fluid dynamics, and the flexibility of the mesh refinement and choosing mesh types through the approximation of the diffusion term using the finite element method. The FE method comprises a reliable technique for the approximation of elliptic problems with complicated geometries. It should be noted that for easier explanation we are using finite volume method in the same meaning as in [13] but are discretizing the domain by a primal mesh and then dual. The approach taken by [13] discretizes the domain in the opposite way.

It is worth noting that, firstly, for double nonlinear models, particularly Richards' equation, fully implicit schemes are preferred over semi-implicit formulations for several reasons. For instance, these schemes are unconditionally stable allowing the performance of larger simulation time steps. Even in the case of discontinuous hydraulic conductivity, the method is stable and robust, see e.g. [6]. Secondly, approximation of the diffusion using mixed finite element method entails approximating the convection part using cell centered finite volume.

We study the monotonicity and convergence analysis of our full implicit scheme for the problem (1.1), based on suitable assumptions, to the weak solution. For this purpose, a priori estimates are carried out based on several previous works by Eymard et al. [13] and Mahmood [17], see also [16,15], and the Kolmogorov compactness theorem. This analysis may be of interest to prove a convergence of higher order FV–FE scheme for which the flux is nonlinear as well, this will be shown in a future work, see [19,23]. It is worth noting that order of convergence has been shown for double nonlinear models in previous studies [2,26]. Existence of a unique weak solution for our model has been studied intensively, see e.g. [1,20] and the references therein.

This paper is organized as follows: the discretization by a relaxation scheme with a combined finite volume–finite element methods is given in Section 2. Section 3 is devoted to derive sufficient conditions to obtain a monotonic scheme and