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# An unconditionally stable hybrid method for image segmentation

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ABSTRACT

In this paper, we propose a new unconditionally stable hybrid numerical method for minimizing the piecewise constant Mumford–Shah functional of image segmentation. The model is based on the Allen–Cahn equation and an operator splitting technique is used to solve the model numerically. We split the governing equation into two linear equations and one nonlinear equation. One of the linear equations and the nonlinear equation are solved analytically due to the availability of closed-form solutions. The other linear equation is discretized using an implicit scheme and the resulting discrete system of equations is solved by a fast numerical algorithm such as a multigrid method. We prove the unconditional stability of the proposed scheme. Since we incorporate closed-form solutions and an unconditionally stable scheme in the solution algorithm, our proposed scheme is accurate and robust. Various numerical results on real and synthetic images with noises are presented to demonstrate the efficiency, robustness, and accuracy of the proposed method.


## 1. Introduction

Image segmentation is one of the fundamental tasks in automatic image analysis. Its goal is to partition a given image into regions that contain distinct objects. For example, the segmentation of structures from images is an important first step for object recognition [6], interpretation [21], image restoration [29], and image inpainting [4,7,9]. The most common form of segmentation is based on the assumption that distinct objects in an image have different and approximately constant colors. A natural approach is therefore to decompose an image domain into approximately homogeneous regions that are separated by sharp changes in image features. One of the general approaches for image segmentation is the minimizer of the piecewise constant Mumford–Shah functional [23]. Chan–Vese [10,28] solved the minimization problem by the level set method proposed by Osher and Sethian [24]. Recently, the Allen–Cahn equation [1] has been used in image segmentation [3,8,14,18,17]. In particular, Esedoḡlu and Tsai [14] used the Allen–Cahn equation to solve the reduced Mumford–Shah problem with the Chan–Vese fitting terms.

In this paper, we propose an unconditionally stable hybrid numerical method which consists of the Allen–Cahn equation and a fitting term. An operator splitting technique is used to solve the model numerically. We describe its numerical solution algorithm and give a proof of the unconditional stability of the scheme. We also present various numerical results on real

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and synthetic images with various types and levels of noise to demonstrate the efficiency, robustness, and accuracy of the proposed numerical method.

This paper is organized as follows. In Section 2, three models for image segmentation are briefly reviewed. In Section 3, we describe the proposed unconditionally stable hybrid operator splitting method and provide a proof of the unconditional stability of the scheme. In Section 4, we perform some characteristic numerical experiments for image segmentation. Finally, conclusions are given in Section 5.

## 2. Description of the previous models

In this section, we briefly review three approaches such as Mumford–Shah, Chan–Vese, and phase-field models for image segmentation.

### 2.1. Mumford–Shah model

With a given image $f_0$ on the image domain $\Omega$ and its segmenting curve $C$, Mumford and Shah [23] proposed that the segmentation of an image can be obtained through the minimization of the following Mumford–Shah energy functional:

$$\mathcal{E}^{\mathrm{MS}}(f, C) = \mu Length(C) + \int_{\Omega} \left|f_0(\mathbf{x}) - f(\mathbf{x})\right|^2 d\mathbf{x} + \nu \int_{\Omega\backslash C} \left|\nabla f(\mathbf{x})\right|^2 d\mathbf{x},$$

where $\mu$ and $\nu$ are positive parameters and $f$ is the piecewise smooth approximation to $f_0$. However, in practice it is not easy to minimize this functional because of the unknown set $C$ of lower dimension than $f$.

### 2.2. Chan–Vese model

Chan and Vese [10] proposed an algorithm for decomposing the image into two regions with piecewise constant approximations by minimizing the energy of the Mumford and Shah functional

$$\mathcal{E}^{\mathrm{CV}}(c_1, c_2, C) = \mu Length(C) + \lambda_1 \int_{inside(C)} \left|f_0(\mathbf{x}) - c_1\right|^2 d\mathbf{x} + \lambda_2 \int_{outside(C)} \left|f_0(\mathbf{x}) - c_2\right|^2 d\mathbf{x},$$

where $\mu$, $\lambda_1$, and $\lambda_2$ are positive parameters [15,20]. The constants $c_1$ and $c_2$ are the averages of $f_0$ inside and outside of $C$, respectively. Chan and Vese replaced the unknown curve $C$ by the level-set function $\phi(\mathbf{x})$. Then the energy functional $\mathcal{E}^{\mathrm{CV}}(c_1, c_2, C)$ can be rewritten as

$$\begin{aligned}\mathcal{E}^{\mathrm{CV}}(c_1, c_2, \phi) = {} & \mu \int_{\Omega} \delta_\epsilon\big(\phi(\mathbf{x})\big)\left|\nabla\phi(\mathbf{x})\right| d\mathbf{x} + \lambda_1 \int_{\Omega} \left|f_0(\mathbf{x}) - c_1\right|^2 H_\epsilon\big(\phi(\mathbf{x})\big)\, d\mathbf{x} \\ & + \lambda_2 \int_{\Omega} \left|f_0(\mathbf{x}) - c_2\right|^2 \big(1 - H_\epsilon\big(\phi(\mathbf{x})\big)\big)\, d\mathbf{x}.\end{aligned}$$

By applying the gradient descent method, we obtain the following equation:

$$\frac{\partial\phi}{\partial t} = \delta_\epsilon(\phi)\left[\mu\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - \lambda_1\big(f_0(\mathbf{x}) - c_1\big)^2 + \lambda_2\big(f_0(\mathbf{x}) - c_2\big)^2\right].$$

The level set based algorithm of Chan and Vese can be used to process the image with a large amount of noise and detect objects whose boundaries cannot be defined by gradient. For more details about parameters and description of equations, please refer to Ref. [10].

### 2.3. Phase-field model

A phase-field approximation for minimizing the Mumford–Shah functional, by using the Allen–Cahn equation to replace the length of the segmenting curve $C$, is given by the following energy functional:

$$\mathcal{E}(\phi) = \int_{\Omega} \left(\frac{F(\phi)}{\epsilon^2} + \frac{|\nabla\phi|^2}{2} + G(\phi, f_0)\right) d\mathbf{x}, \tag{1}$$

where $F(\phi) = 0.25(\phi^2 - 1)^2$ is a double-well potential as shown in Fig. 1, $\epsilon$ is the gradient energy coefficient related to the interfacial energy, and $\Omega$ is the image domain.

When $\phi$ is locally equilibrated, the first two terms in Eq. (1) are proportional to the length of the segmenting curve $C$ [12] by