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A very fast and accurate boundary element method for options with moving barrier and time-dependent rebate

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ABSTRACT

A numerical method to price options with moving barrier and time-dependent rebate is proposed. In particular, using the so-called Boundary Element Method, an integral representation of the barrier option price is derived in which one of the integrand functions is not given explicitly but must be obtained solving a Volterra integral equation of the first kind. This equation is affected by several kinds of singularities, some of which are removed using a suitable change of variables. Then the transformed equation is solved using a low-order finite element method based on product integration. Numerical experiments are carried out showing that the proposed method is extraordinarily fast and accurate. In particular a high level of accuracy is achieved also when the initial price of the underlying asset is close to the barrier, when the barrier and the rebate are not differentiable functions, or when the option's maturity is particularly long.

1. Introduction

In this paper we propose and test a Boundary Element Method to price options with (single) moving barrier and time-dependent rebate on an underlying described by the Black–Scholes model.

Barrier options with time-dependent barrier are largely traded in over-the-counter markets (see [10]). Nevertheless pricing options with time-dependent barrier constitutes an interesting problem also because it allows to price barrier options in the frequent case where the parameters of the Black–Scholes model (interest rate and volatility) are assumed to be time-dependent. In fact, using a simple change of variables, the problem of pricing an option with moving barrier and time-dependent parameters can be transformed into the problem of pricing an option with moving barrier and constant parameters (see for example [14]).

The Boundary Element Method (BEM), originally developed by Brebbia, Dominguez, Banerjee and Butterfield [3,6], is a numerical method to solve partial differential equations. Its main advantage is that it allows to reduce a partial differential equation defined on a whole domain to a boundary integral equation (see [7]).

Generally speaking a barrier option is an option whose payoff depends on whether or not the price of the underlying asset breaches some barrier level during the option's lifetime. Barrier options are grouped in two main categories: *in* options, that come into existence when the underlying asset price breaches the barrier, and *out* options, that become worthless if the underlying asset price breaches the barrier prior to maturity. In addition barrier options are said *down* if the barrier is triggered when crossing from below, or *up* if the barrier is triggered when crossing from above. Barrier options can also pay a rebate: in particular the holder of an out option receives a predetermined amount of money if the barrier is breached,

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whereas the holder of an in option receives a predetermined amount of money if the barrier is never crossed during the option's lifetime. Finally we mention the case of double barrier options, where the payoff depends on the breaching behaviors of the underlying asset price with respect to two barrier levels.

Exact analytical formulae to price various kinds of barrier options (see [13,15,16]) have been derived in the case where the barrier function is exponentially affine and the underlying asset price follows the Black–Scholes dynamics with constant parameters (interest rate and volatility). However, if either the barrier is not exponentially affine or the parameters are assumed to be time-dependent, then exact closed-form solutions can no longer be obtained, and numerical approximations are required.

Barrier options are often priced using binomial and trinomial trees (see for example [1,4,5,17–19]), which, as noted by [21], can be regarded as applications of the explicit finite difference method. However the rates of convergence obtained using these approaches are not particularly high, especially when the underlying asset price is close to the barrier.

In the present paper, using the BEM, an integral representation of the barrier option price is derived in which one of the integrand functions is not given explicitly but must be obtained solving a Volterra integral equation of the first kind. Such an integral equation contains several kinds of singularities, some of which are removed using a suitable change of variables. Then the transformed equation is solved using a low-order finite element method based on product integration.

Several numerical experiments are presented showing that the proposed method is extraordinarily accurate and fast. In fact, if the simulation is carried out on a computer with a Pentium III processor, relative errors of order 10^{-6} and 10^{-7} are usually obtained in times of order 10^{-1} s and 10^{-2} s. Excellent results are also experienced when the initial price of the underlying asset is very close to the barrier (the distance of the initial price of the underlying asset from the barrier is smaller than one percentage point), when the barrier and the rebate are not differentiable functions, or when the option's maturity is particularly long (see Section 6).

In addition, as highlighted by the simulation of a benchmark test-case, the numerical method presented in this paper significantly outperforms the improved lattices proposed in [4,5,17,18].

We may note that the numerical method developed in this paper shares some similarities with an approach sketched in [11]. Precisely, in that work, using the method of double layer potentials, an integral formulation of the barrier option price is obtained in which an integral weight function is unknown and must be determined solving a Volterra integral equation of the second kind (Eq. (12.14) in [11]). Such an integral equation (which is different from the one derived in the present paper) is singular and a method to regularize its integral kernel is suggested based on the first-order Taylor series expansion of the barrier function.

Nevertheless, the above approach has some drawbacks: first of all the proposed regularization procedure is not completely effective when dealing with a Call or a Put barrier option (note that in [11] only the case of an option with constant payoff is considered in detail, whereas the case of Call and Put options is just sketched). In fact, in the case of a Call or a Put option, not only the integral kernel, but also the right-hand side of the Volterra integral equation is a non-differentiable function (see [11], p. 468), and thus it is no longer sufficient to use a regularization procedure which acts on the integral kernel only. Furthermore, in [11] the singularity of the integral formulation that gives the option price is not handled efficiently. In fact, in dealing with such a singularity, the unknown weight function is assumed to be constant on a certain time interval close to the option's maturity, whereas in reality it is not (furthermore no suggestion on how to select such a time interval is provided).

Speaking more generally we can say that the approach sketched in [11] does not take into account the asymptotic behavior of the integral formulation at times close to the option's maturity. By contrast the numerical scheme developed in the present paper is based on the asymptotic analysis of the integral formulation at times close to the option's maturity (see Section 4) and it is right thanks to this fact that we achieve excellent results. Finally we point out that the main contribution of the present paper is not only to derive an integral formulation of the barrier option problem, but also to develop an efficient approach to regularize and solve the Volterra integral equation (thus showing that the BEM integral approach is particularly suitable for barrier option pricing). By contrast in [11] no method to approximate the Volterra integral equation is proposed, and no numerical results are presented.

We also make notice that another advantage of the BEM integral formulation presented in this paper is that it is rather straightforward to obtain and can be easily extended to models of asset price dynamics other than the Black–Scholes model (see for example [2]).

The remainder of this paper is organized as follows: in the next section we briefly review the mathematical model that allows to price options with moving barrier and time-dependent rebate; in Section 3 we present the BEM integral formulation of the partial differential problem described in Section 2; in Section 4 and Section 5 we develop the numerical method to solve the Volterra integral equation; in Section 6 we show and discuss the results obtained using the proposed numerical method; finally in Section 7 some conclusions are drawn.

2. Mathematical preliminaries

For the sake of clarity the numerical method will be presented directly on the case of up and out call options. However the reader will note that the algorithm described in the next section can also be used, with minor modification, to price up and out put options, down and out call options, and down and out put options. Moreover we recall that the price of *in* options can be obtained from the price of *out* options using simple in–out parity relations (see [23]).