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A fourth-order approximate projection method for the incompressible Navier–Stokes equations on locally-refined periodic domains

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ABSTRACT

In this follow-up of our previous work [30], the author proposes a high-order semi-implicit method for numerically solving the incompressible Navier–Stokes equations on locally-refined periodic domains. Fourth-order finite-volume stencils are employed for spatially discretizing various operators in the context of structured adaptive mesh refinement (AMR). Time integration adopts a fourth-order, semi-implicit, additive Runge–Kutta method to treat the non-stiff convection term explicitly and the stiff diffusion term implicitly. The divergence-free condition is fulfilled by an approximate projection operator. Altogether, these components yield a simple algorithm for simulating incompressible viscous flows on periodic domains with fourth-order accuracies both in time and in space. Results of numerical tests show that the proposed method is superior to previous second-order methods in terms of accuracy and efficiency. A major contribution of this work is the analysis of a fourth-order approximate projection operator.

1. Introduction

The Navier–Stokes equations are at the center of modern mathematical physics as they govern an enormous range of common phenomena such as ocean currents, blood flow, the atmosphere, and turbulence. For incompressible fluids, the non-dimensional form of these equations is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{g} - \nabla p + \nu\Delta\mathbf{u}, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

where $t \in \mathbb{R}$ is time, $\mathbf{x} \in \mathbb{R}^{\mathrm{D}}$ ($\mathrm{D} = 2, 3$) the spatial location, $\mathbf{g}$ the external forcing term, p the nondimensionalized pressure, $\mathbf{u}$ the velocity, $\nu = 1/\mathrm{Re}$, and Re the Reynolds number. (1) is also known as the incompressible Navier–Stokes equations (INSE).

Among the numerous computational methods for the INSE, the projection method introduced by Chorin [9] first computes an auxiliary velocity field $\mathbf{u}^*$ from the momentum equation by ignoring the pressure gradient term and then project $\mathbf{u}^*$ onto the divergence-free space to fulfill the incompressibility constraint. During the last 45 years, many variants of the original projection method have been proposed, some successful examples are those of Kim and Moin [19], Bell et al. [5], Orszag et al. [24], and E and Liu [12].

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One attractive feature of the projection methods is its efficiency: during one time step it is only necessary to solve a sequence of elliptic equations either for the velocity or for the pressure. Alternative, one can solve for the velocity and the pressure simultaneously by forming the linear system as a saddle point problem [6]. For this approach, Griffith [14] proposes to use the projection method as a preconditioner instead of a solver. Indeed, the linear system after the preconditioning has extremely clustered eigenvalues and can be solved by a Krylov subspace method within a few iterations. One advantage of this approach is the ease of imposing a variety of physical boundary conditions.

In comparison to second-order projection methods, the literature on fourth-order methods is much thinner; see, e.g., [17, 16]. There also exist high-order finite difference/volume methods [15,25] that do not belong to the class of projection methods. However, most of these methods have explicit time integration, which incurs unnecessary restriction on the time-step size from the diffusion term. Shukla et al. [26] coupled very high order spatial discretizations with a second-order semi-implicit time integrator, the overall accuracy for transient flows is still of second order. It is therefore desirable to have a high-order method of which the spatial and temporal accuracies are comparable and of which the time-step size is not constrained by the diffusion term.

In this paper the author proposes such a method for solving INSE with fourth-order accuracies both in time and in space, based on our previous work on the advection–diffusion equation [30]. In the proposed method, spatial discretization employs classical finite-volume stencils while the temporal integration adapts a semi-implicit additive Runge–Kutta (ARK) method [18], which admits explicit treatment of the non-stiff convection term and implicit treatment of the stiff diffusion term. As such, the time-step size is only constrained by the convection term. To fulfill the incompressibility constraint, an approximate projection operator is analyzed for its stability and accuracy. On periodic domains, combining these components yields a straightforward algorithm, thanks to the commutativity of the discrete Laplacian and projection operators. The flexibility and efficiency of the proposed method is further enhanced by implementing it in the context of structured adaptive mesh refinement (AMR). One significance of the AMR implementation is that the proposed method cannot be substituted by another high-order spectral method.

The rest of this paper is organized as follows. Section 2 contains fourth-order finite-volume discretization of various operators. In particular, the fourth-order approximate projection operator is analyzed for its stability and accuracy in Section 2.2 and single-level discrete operators are generalized to multiple-level AMR hierarchies in Section 2.3. In Section 3, the fourth-order method for solving the INSE on periodic domains is proposed based on the IMEX scheme, a particular type of the ARK method. In Section 4, two benchmark problems are numerically simulated using the proposed fourth-order method, showing that it is superior to a second-order method in terms of efficiency and accuracy. Section 5 concludes this paper with a prospect of future research.

2. Spatial discretization

The rectangular problem domain Ω is discretized into a collection of rectangular grid cells. Denote a grid cell by a multi-index $\mathbf{i} \in \mathbb{Z}^{\mathrm{D}}$, its region is represented by

$$\mathcal{C}_{\mathbf{i}} = \left[\mathbf{x}_O + \mathbf{i}h,\ \mathbf{x}_O + (\mathbf{i}+\mathbb{1})h\right], \tag{2}$$

and the region of the higher face of cell $\mathbf{i}$ in dimension d by

$$\mathcal{F}_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \left[\mathbf{x}_O + \left(\mathbf{i}+\mathbf{e}^d\right)h,\ \mathbf{x}_O + (\mathbf{i}+\mathbb{1})h\right], \tag{3}$$

where $\mathbf{x}_O \in \mathbb{R}^{\mathrm{D}}$ is some fixed origin of the coordinates, h the uniform mesh spacing, $\mathbb{1} \in \mathbb{Z}^{\mathrm{D}}$ the multi-index with all its components equal to one, and $\mathbf{e}^d \in \mathbb{Z}^{\mathrm{D}}$ a multi-index with 1 as its d-th component and 0 otherwise. Throughout this paper, regular lowercase letters denote scalars, boldface lowercase letters denote vectors, and boldface uppercase letters denote operators.

The author strictly distinguishes three different types of quantities, viz. point values, cell averages, and face averages. Point values are denoted by naked symbols with the subscripts indicating their locations, e.g. $\phi_{\mathbf{i}}$ and $\phi_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}$ are the values of ϕ at the centers of the cell $\mathbf{i}$ and the face $\mathbf{i}+\frac{1}{2}\mathbf{e}^d$, respectively. A symbol within a pair of angle brackets is a cell/face average if the subscript is an integer/fraction. For example, the averaged ϕ over cell $\mathbf{i}$ is denoted by

$$\langle\phi\rangle_{\mathbf{i}} = \frac{1}{h^{\mathrm{D}}} \int_{\mathcal{C}_{\mathbf{i}}} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{4}$$

and the averaged ϕ over the face $\mathbf{i}+\frac{1}{2}\mathbf{e}^d$ by

$$\langle\phi\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \frac{1}{h^{\mathrm{D}-1}} \int_{\mathcal{F}_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{5}$$