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A new Jacobi rational–Gauss collocation method for numerical solution of generalized pantograph equations

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ABSTRACT

This paper is concerned with a generalization of a functional differential equation known as the pantograph equation which contains a linear functional argument. In this article, a new spectral collocation method is applied to solve the generalized pantograph equation with variable coefficients on a semi-infinite domain. This method is based on Jacobi rational functions and Gauss quadrature integration. The Jacobi rational-Gauss method reduces solving the generalized pantograph equation to a system of algebraic equations. Reasonable numerical results are obtained by selecting few Jacobi rational–Gauss collocation points. The proposed Jacobi rational–Gauss method is favorably compared with other methods. Numerical results demonstrate its accuracy, efficiency, and versatility on the half-line.

1. Introduction

Over the last three decades, the scientists have paid much attention to spectral method due to its high accuracy (see for instance, [11,5,14,15,42] and the references therein). Many problems in science and engineering arise in unbounded domains. Several spectral approximations for ordinary differential equations in unbounded domains have been investigated by many authors (see, e.g. [19,20,7]). Chebyshev rational and Legendre rational functions are particular cases of the Jacobi rational functions. These functions have been successfully applied in both the solution of boundary value problems on the semi-infinite interval and in computational fluid dynamics [21,22,31,32]. In general, the use of Jacobi rational functions ($R_j^{(\alpha,\beta)}(x)$ with $\alpha,\beta\in(-1,\infty)$, $x\in(0,\infty)$ and j is the polynomial degree) has the advantage of obtaining the solutions of boundary value problems in terms of the Jacobi rational indexes α and β (see, e.g., [21,22]). Therefore, instead of proposing spectral approximation results for each particular pair of Jacobi rational indexes, it would be very useful to carry out a systematic study on Jacobi rational functions with general indexes α,β which can then be directly applied to other applications.

In order to use the spectral approximations to the problems in unbounded domains, it is often necessary to introduce the rational approximations. In [6], Boyd introduced a new orthogonal functions which are mutually orthogonal systems, named Chebyshev rational functions on the half-line, by mapping to Chebyshev polynomials. Authors of [21,22] proposed Legendre

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and Chebyshev rational spectral approximations which are based on weighted orthogonal systems consisting of rational functions built from Legendre and Chebyshev polynomials under a rational transformation, respectively. In fact, it is shown in [21,22] that the Legendre and Chebyshev rational approximations are the efficient spectral methods for problems in the semi-infinite intervals. Boyd et al. [8] proposed the spectral collocation methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine. Moreover, Parand et al. [31,32] derived the operational matrices of derivatives of Chebyshev and Legendre rational functions for solving some problems in semi-infinite intervals. Recently, Yi and Guo [42] analyzed and developed the generalized Jacobi rational spectral method for solving partial differential equations of degenerate type.

Delay differential equations have a wide range of application in science and engineering. Functional differential equations with proportional delays are usually referred as pantograph equations or generalized pantograph equations. The name pantograph originated from the study work of Ockendon and Tayler [29]. The applications of pantograph equations are in different fields such as number theory, economy, biology, control, electrodynamics, nonlinear dynamical systems, quantum mechanics, probability theory, astrophysics, cell growth and other industrial applications. For some applications of this equation we refer the interested reader to [29,1,9]. Properties of the analytic solution of this equation as well as numerical methods have been studied by several authors [18,34,27,38].

The pantograph equation is a kind of delay differential equations. Pantograph equation was studied by many authors numerically and analytically. Sezer et al. [34] obtained the approximate solution of multi-pantograph equation with non-homogeneous terms using Taylor polynomials. In [37] the authors applied the Taylor method to approximate solution of the non-homogeneous multi-pantograph equation with variable coefficients, which is extending of the multi-pantograph equation given in [27]. Li and Liu [25] gave the sufficient condition that assure the Runge–Kutta methods with a variable step size are asymptotically stable when it is applied to the multi-pantograph equation. Alomari et al. [2] used homotopy analysis method to solve a class of delay differential equations. Recently, the authors in [43,33] developed the Variational iteration method to solve the multi-pantograph delay and the generalized pantograph equations. Moreover, the Bessel and Hermite polynomials are used to obtain the approximation solution of generalized pantograph equation with variable coefficients in [44] and [41], respectively. More recently, Tohidi et al. [39] proposed Bernoulli collocation method based on operational matrix of derivatives for solving generalized pantograph equation with variable coefficients.

In this manuscript, we propose a Jacobi rational–Gauss collocation method to solve numerically the generalized pantograph equation with variable coefficients [41]:

$$u^{(m)}(x)=\sum_{j=0}^{J}\sum_{n=0}^{m-1}p_{jn}(x)u^{(n)}(\lambda_{jn}x+\mu_{jn})+g(x),\quad x\in(0,\infty),\tag{1.1}$$

which is the generalization of the pantograph equation given in [13,17,36], subject to the mixed initial conditions

$$\sum_{n=0}^{m-1}a_{in}u^{(n)}(0)=\lambda_i,\quad i=0,1\ldots m-1,\tag{1.2}$$

where a_{in}, λ_i and λ_{jn} are real or complex coefficients, meanwhile $p_{jn}(x)$ and $g(x)$ are given continuous functions in the interval $[0,T]$.

The main aim of this manuscript is to propose a new method to approximate the generalized pantograph equation on a semi-infinite interval using the Jacobi rational functions. We propose the spectral Jacobi rational–Gauss collocation (JRC) method to find the solution $u_N(x)$. For suitable collocation points we utilize the $(N-m+1)$ nodes of the Jacobi rational–Gauss interpolation on $(0,\infty)$. These equations together with initial conditions generate system of $(N+1)$ algebraic equations which can be solved. Numerical results are exhibiting the usual exponential convergence behavior of spectral approximations.

This manuscript is organized as follows. In Section 2, we introduce the Jacobi rational functions and their relevant properties needed hereafter, and in Section 3, the way of constructing the collocation method for solving generalized pantograph equation is implemented using the Jacobi rational functions. The numerical results that exhibiting the accuracy and efficiency of our proposed spectral algorithms are introduced in Section 4. A conclusion is given in Section 5.

2. Jacobi rational interpolation

In this section, we detail the properties of Jacobi polynomials and Jacobi rational functions that will be used to construct the JRC method.

2.1. Jacobi polynomials

The Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$, $k=0,1,2,\ldots$, are the eigenfunctions of the Sturm–Liouville problem

$$\partial_y\big((1-y)^{\alpha+1}(1+y)^{\beta+1}\partial_y \upsilon(y)\big)+\lambda(1-y)^{\alpha}(1+y)^{\beta}\upsilon(y)=0,\quad y\in I=[-1,1].\tag{2.1}$$

Their corresponding eigenvalues are $\lambda_k^{(\alpha,\beta)}=k(k+\alpha+\beta+1)$, $k=0,1,2,\ldots$.