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Efficient and stable perfectly matched layer for CEM

Kenneth Duru [a], Gunilla Kreiss [b]

[a] Department of Geophysics, Stanford University, Stanford, CA, United States
[b] Division of Scientific Computing, Uppsala University, Uppsala, Sweden

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ABSTRACT

An efficient unsplit perfectly matched layer for numerical simulation of electromagnetic waves in unbounded domains is derived via a complex change of variables. In order to surround a Cartesian grid with the PML, the time-dependent PML requires only one (scalar) auxiliary variable in two space dimensions and six (scalar) auxiliary variables in three space dimensions. It is therefore cheap and straightforward to implement. We use Fourier and energy methods to prove the stability of the PML. We extend the stability result to a semi-discrete PML approximated by central finite differences of arbitrary order of accuracy and to a fully discrete problem for the 'Leap-Frog' schemes. This makes precise the usefulness of the derived PML model for longtime simulations. Numerical experiments are presented, illustrating the accuracy and stability of the PML.

1. Introduction

The perfectly matched layer (PML) – since the introduction [9] – is now a standard approach to truncate unbounded spatial domains in computational electromagnetics (CEM). In its original form [9], the PML is derived by splitting the wave functions (magnetic and electric fields) before special lower order terms that simulate the absorption of waves are added. This makes it possible for the PML to absorb all waves exiting the computational domain without reflections, independent of angles of incidence and frequencies. There are other formulations of the PML which do not require the unphysical splitting of the field variables, see for instance [13,5]. These are the so-called unsplit PML. The unsplit PML is more elegant mathematically. It is also more straightforward to be implemented in a computer program. The PML, both split [9] and unsplit [13,18,5], have also been extended to many application areas like acoustics, elasticity and quantum dynamics.

Despite the general success of the PML in CEM, there are still some challenges associated with the PML. One of the drawbacks of the PML for hyperbolic problems in the time domain is that extra (auxiliary) variables and equations are needed to guarantee the perfect matching property. A direct application of the modal PML to the Maxwell's equations in three (two) space dimensions requires about eighteen (six) auxiliary variables to surround a computational domain with the PML. If instead one adopts the approach in [15,5] and add the PML to only the normal propagating fields, at least twelve (four) auxiliary variables are needed to surround a computational domain with the PML in three (two) space dimensions. The extra variables imply that extra memory and computational time are needed to store and update the auxiliary variables (or compute convolution operations). Often one can argue that the extra memory and computational time required are usually counterbalanced by reducing the size of the computational domain due to the perfect matching property. However, efficiency can be increased by decreasing the number of the auxiliary variables and equations in the layer as far as possible, while preserving the perfect matching property.

For hyperbolic systems that naturally come in second order formulation, like the standard wave equation, it has been demonstrated that the number of auxiliary variables needed in the layer can be minimized by deriving the PML directly

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for the second order system, [14,12]. However, rewriting the Maxwell's equations as a second order system leads to a coupled vector wave equation for the electric and magnetic fields respectively. In [14], an efficient PML was constructed for the scalar wave equation with minimal number of auxiliary variables. For the scalar wave equation, in order to surround a computational domain with the PML, the formulation [14] requires four (two) auxiliary variables in three (two) space dimensions. This implies that the PML for the second order formulation of the Maxwell's equations in three space dimensions will require at the least twenty four (twelve for each field variable) auxiliary variables to surround a computational cube with the PML. In this paper, we will show that it is possible to construct a PML directly for the first order system and carefully choose auxiliary variables such a smaller number of auxiliary variables and equations are needed.

The Maxwell's equations are described by a symmetric hyperbolic system. This implies that the solutions (electric and magnetic fields) in the absence of boundaries are uniformly bounded in time by the initial data. On the other hand, the split-field PML and also many unsplit PML are only weakly well-posed. It is well-known that a weakly well-posed problem may become ill-posed when a lower order term is added. It was demonstrated in [1] that under certain lower order perturbations the PML for Maxwell's equations can support explosive modes, making numerical computations impracticable. Because of the result [1] several strongly well-posed unsplit PML were developed [2]. However, for linear problems, in several studies numerical computations have shown that the lack of strong well-posedness for the PML may not be fatal. It was indeed shown in [7] that the lower order terms appearing in the PML for Maxwell's equations cannot lead to an exponential growth in the PML. The study of the temporal stability properties of the PML in unbounded domains is a topic of several works such as in [3,4,7,6,8,12] and many others.

In the present paper, we construct an unsplit PML for Maxwell's equations using the complex coordinate stretching technique. In the Laplace space our PML is equivalent to the standard unsplit PML [13,5]. In order to localize the PML in time we judiciously choose auxiliary variables, and then invert the Laplace transforms. Our choice of auxiliary variables reduces dramatically the number of auxiliary variables and equations needed in the layer. We note that for the transverse magnetic (TM_z) case in two space dimensions, only one scalar auxiliary variable is needed to surround the computational domain with the PML. The Maxwell's equations in three space dimensions require only six scalar auxiliary variables. The main contribution of this paper is that we can minimize the number of auxiliary variables needed in the PML while preserving the mathematical properties of the layer. A detailed mathematical analysis of the PML is also presented. We use Fourier and energy methods to prove the asymptotic stability of the PML. We extend the stability results to a semi-discrete PML approximated by central finite difference operators of arbitrary order of accuracy, and to a fully discrete problem for a modified 'Leap-Frog' scheme. Numerical experiments are presented, illustrating the accuracy and asymptotic stability of the PML.

The remainder of the paper is planned as follows. In the next section, we recall the Maxwell's equations in two space dimensions and introduce the PML equations. We use Fourier analysis and energy methods to prove well-posedness and stability of the PML. In Section 3, we extend the stability results to discrete approximations of the PML equations. We present some numerical examples in Section 5. A brief conclusion and suggestions for future work are offered in the last section. In Appendix A we present the PML equations for the Maxwell's equations in three space dimensions.

2. The perfectly matched layer

We begin with the TM_z case of the Maxwell equation in a source free, homogeneous isotropic medium,

$$\begin{aligned} \frac{\partial E_z}{\partial t} &= -\frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y}, \\ \frac{\partial H_y}{\partial t} &= -\frac{\partial E_z}{\partial x}, \\ \frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y}. \end{aligned} \tag{1}$$

Here

$$\mathbf{H} = (H_x, H_y, 0)^T, \qquad \mathbf{E} = (0, 0, E_z)^T,$$

are the magnetic fields and electric fields respectively. Assume that we are interested in the numerical solutions of the Maxwell's equation (1) in the computational domain $\Omega = [-x_0, x_0] \times [-y_0, y_0]$, $x_0, y_0 > 0$. Outside Ω, that is in $\mathbb{R}^2 \setminus \Omega$, we introduce the PML such that all waves leaving the computational domain are absorbed without reflections. The PML equations for (1), completely surrounding the computational domain Ω, is

$$\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial t} + \sigma_1(x)\mathcal{E}_z &= -\frac{\partial \mathcal{H}_y}{\partial x} + \frac{\partial \mathcal{H}_x}{\partial y} + \mathcal{H}_x^*, \\ \frac{\partial \mathcal{H}_y}{\partial t} + \sigma_1(x)\mathcal{H}_y &= -\frac{\partial \mathcal{E}_z}{\partial x}, \\ \frac{\partial \mathcal{H}_x}{\partial t} + \sigma_2(y)\mathcal{H}_x &= \frac{\partial \mathcal{E}_z}{\partial y}, \\ \frac{\partial \mathcal{H}_x^*}{\partial t} + \sigma_2(y)\mathcal{H}_x^* &= \big(\sigma_1(x) - \sigma_2(y)\big)\frac{\partial \mathcal{H}_x}{\partial y}. \end{aligned} \tag{2}$$