Applied Numerical Mathematics 76 (2014) 93–100

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A trust region method for constructing triangle-mesh approximations of parametric minimal surfaces

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ARTICLE INFO

Article history:
Received 31 December 2012
Received in revised form 14 July 2013
Accepted 16 October 2013
Available online 31 October 2013

Keywords:
Minimal surface
Triangle mesh
Trust region
Variable metric method

ABSTRACT

Given a function f_0 defined on the unit square Ω with values in $\mathbb{R}^3$, we construct a piecewise linear function f on a triangulation of Ω such that f agrees with f_0 on the boundary nodes, and the image of f has minimal surface area. The problem is formulated as that of minimizing a discretization of a least squares functional whose critical points are uniformly parameterized minimal surfaces. The nonlinear least squares problem is treated by a trust region method in which the trust region radius is defined by a stepwise-variable Sobolev metric. Test results demonstrate the effectiveness of the method.

1. Introduction

In [11] we introduced a trust region method in which the trust region radius is measured in the H^1 Sobolev metric. The method was shown to be equivalent to a method which blends a Newton-like iteration with a steepest descent iteration using the Sobolev gradient and which we termed a Levenberg–Marquardt–Neuberger method. The underlying theory is developed in [3], and the method is applied to the Navier–Stokes equations in [12], and to the Ginzburg–Landau equations in [4]. Here we apply a variation of the method to the problem of approximating minimal surfaces. This research extends the work began in [9]. Refer to [10] for a treatment of the analogous minimal curve length problem.

Other algorithms for constructing triangle-mesh approximations to minimal surfaces have been presented in [1,7,8], and, more recently, in [2] and [6]. By adding a volume constraint to the surface area functional, these methods treat the more general problem of approximating constant mean curvature surfaces, including both soap bubbles with nonzero mean curvature and soap films with zero mean curvature (minimal surfaces). They are thus more powerful than our method but require much more elaborate procedures to maintain stability. More precisely, at least in the case of the Plateau problem with fixed boundary which we treat here, they alternate between updating the geometry as defined by the vertex positions and updating the connection topology by swapping edges. Our method of parameterizing the surface makes it unnecessary to swap edges.

In addition to treating fixed boundaries, the method of [6] also treats free boundaries constrained to lie on a fixed surface or constrained by a fixed length, and nonmanifold boundary curves shared by more than two surface sheets. That method treats the problem of poor mesh quality by replacing the surface area functional by an extension of a centroidal Voronoi tessellation energy functional, where the extension is designed so that minimizing the functional is asymptotically equivalent to minimizing surface area. This results in very high-quality meshes which could be advantageous for estimating differential surface properties. The simpler more efficient method presented here, however, is sufficient for numerical simulations.

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0168-9274/$36.00
http://dx.doi.org/10.1016/j.apnum.2013.10.006

We formulate the continuous problem in Section 2, describe the discretized problem in Section 3, discuss the solution method in Section 4, and present test results in Section 5.

2. Sobolev space formulation

Denote the unit square by $\Omega = [0,1]^2$, and define a parametric representation of regular surfaces by

$$S = \{g \in C^2(\Omega, \mathbb{R}^3): g_1 \times g_2 \neq \mathbf{0}\},$$

where g_1 and g_2 denote first partial derivatives. Note that S does not include the zero function and is therefore not a linear space. Rather, it is an infinite-dimensional manifold which will be equipped with a Riemannian metric in the form of an f-dependent Sobolev inner product in the tangent space at each point $f \in S$. The surface area functional is

$$\psi(f) = \int_\Omega |f_1 \times f_2|$$

for $f \in S$, where $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^3$. Now fix $f_0 \in S$ and define the linear space of compactly supported variations

$$S_0 = \{h \in C^2(\Omega, \mathbb{R}^3): h_{|\partial\Omega} = 0\},$$

where $\partial\Omega$ denotes the boundary of Ω. A minimal surface is obtained by minimizing ψ over functions f that agree with f_0 on the boundary of Ω:

$$\psi'(f)h = \lim_{\alpha \to 0} \frac{1}{\alpha}\left[\psi(f + \alpha h) - \psi(f)\right] = 0$$

for all $h \in S_0$. More precisely, a minimal surface is the image of a critical point of ψ.

Consider the least squares functional

$$\phi(f) = \frac{1}{2}\int_\Omega |f_1 \times f_2|^2.$$

This is similar in appearance but not to be confused with the Dirichlet integral of f. It was shown in [9] that critical points of ϕ are critical points of ψ with $f_1 \times f_2$ constant, so that the critical points are uniformly parameterized, and critical points of ψ are, with a change of parameters, critical points of ϕ. In the case of minimizing curve length, the analogue of ϕ has critical points corresponding to constant-velocity trajectories with zero curvature. Our computational procedure involves a discretization of ϕ rather than of ψ. This has an enormous advantage in terms of avoiding degenerate triangles and the ill-conditioning associated with widely varying triangle areas.

In order to simplify the derivation of expressions for gradients and Hessians of ϕ, define a nonlinear differential operator A by

$$A(f) = f_1 \times f_2$$

for $f \in C^2(\Omega, \mathbb{R}^3)$ so that

$$\phi(f) = \frac{1}{2}\langle A(f), A(f)\rangle_{L^2},$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the standard inner product associated with $L^2(\Omega, \mathbb{R}^3)$. Note that, since f has continuous mixed second partial derivative $f_{12} = f_{21}$, $A'(f)$ is self-adjoint on S_0 in the L^2 inner product:

$$\begin{aligned}\langle A'(f)h, g\rangle_{L^2} &= \int_\Omega \langle f_1 \times h_2 + h_1 \times f_2, g\rangle = \int_\Omega \langle g \times f_1, h_2\rangle + \langle h_1, f_2 \times g\rangle = \int_\Omega -\langle (g \times f_1)_2, h\rangle - \langle h, (f_2 \times g)_1\rangle \\ &= \int_\Omega \langle h, (f_1 \times g)_2 + (g \times f_2)_1\rangle = \int_\Omega \langle h, f_1 \times g_2 + g_1 \times f_2\rangle = \langle h, A'(f)g\rangle_{L^2}\end{aligned}$$

for all $g, h \in S_0$. While $A(f)$ is not an element of S_0, we can formally apply $A'(f)$ to $A(f)$ to obtain expressions for Fréchet derivatives as follows:

$$\phi'(f)h = \langle A'(f)h, A(f)\rangle_{L^2} = \langle h, [A'(f)]A(f)\rangle_{L^2},$$

and