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Asymptotically exact discontinuous Galerkin error estimates for linear symmetric hyperbolic systems

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ABSTRACT

We present an *a posteriori* error analysis for the discontinuous Galerkin discretization error of first-order linear symmetric hyperbolic systems of partial differential equations with smooth solutions. We perform a local error analysis by writing the local error as a series and showing that its leading term can be expressed as a linear combination of Legendre polynomials of degree p and $p+1$. We apply these asymptotic results to observe that projections of the error are pointwise $\mathcal{O}(h^{p+2})$-superconvergent in some cases. Then we solve relatively small local problems to compute efficient and asymptotically exact estimates of the finite element error. We present computational results for several linear hyperbolic systems in acoustics and electromagnetism.

1. Introduction

First-order hyperbolic systems arise in many areas of continuum physics when fundamental balance laws are formulated (such as the conservation of mass, momentum, or energy) and if other small-scale, dissipative mechanisms can be neglected. Many of these systems can be written in symmetric form, such as Maxwell's equations of electromagnetism, the wave equation, and the two-dimensional Euler's equation modeling gas dynamics.

The discontinuous Galerkin (DG) finite element method was first used to solve the neutron equation [28] and then studied for initial-value problems for ordinary differential equations [6,27]. Cockburn and Shu [20,19,21] introduced the Runge–Kutta discontinuous Galerkin (RKDG) to solve first-order hyperbolic systems. The solution space of DG methods consists of piecewise continuous polynomial functions. As such, it can sharply capture discontinuities in the solution. They are also locally conservative, and can handle problems with complex geometries to high order. They have a simple communication pattern between elements with a common face, which is useful for parallel computation and adaptive methods, since it is easy to construct locally refined meshes with hanging nodes. Furthermore, they exhibit strong superconvergence that can be used to estimate the discretization error.

A posteriori error estimates are used to guide adaptive algorithms and stop the refinement process. An ideal estimate is (i) *asymptotically correct* in the sense that the error estimate in some norm approaches zero under mesh refinement at the same rate as the actual error and (ii) *computationally efficient* by requiring a small fraction of the solution cost. Several explicit *a posteriori* DG error estimates are known for hyperbolic problems [17,18] where upper bounds of the true error are derived in terms of local residuals and solution jumps. Goal oriented *a posteriori* error estimates have also been derived for hyperbolic systems [24,26]. Explicit error estimates are usually cheaper to use for steering adaptive refinement but can't be relied on to assess the solution quality since, in general, they fail to be asymptotically exact even for smooth solutions.

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Adjerid et al. [6] developed the first asymptotically correct implicit *a posteriori* DG error estimates, based on superconvergence, for one-dimensional linear and nonlinear hyperbolic problems. Later, Adjerid and Massey [8,9] showed how to construct accurate error estimates for multi-dimensional scalar problems on rectangular meshes. They showed that the leading term of error is spanned by two $(p+1)$-degree Radau polynomials in the x and y directions, respectively. Krivodonova and Flaherty [25] showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$. They computed DG error estimates by solving local problems involving numerical fluxes, thus requiring information from neighboring *inflow* elements. Adjerid and Baccouch [2,3] investigated DG methods on structured and unstructured triangular meshes with several finite element spaces to compute accurate error estimates.

Several superconvergence results for DG methods are reported in the literature [2,6,9,7,14,16,15,13]. In [10], we proved that a projection of the DG solution is $O(h^{p+2})$ superconvergent at Radau points and constructed efficient *a posteriori* error estimates. However, our proofs were valid for special linear symmetric hyperbolic systems in two space dimensions satisfying the assumptions of Lemma 3.2 [10], for which either (i) at least one of the coefficient matrices $\mathbf{A}_1$ or $\mathbf{A}_2$, are invertible, or (ii) $\mathcal{N}(\mathbf{P}_{1,2}^t\mathbf{A}_2\mathbf{P}_{1,2})=\{0\}$ or $\mathcal{N}(\mathbf{P}_{2,2}^t\mathbf{A}_1\mathbf{P}_{2,2})=\{0\}$, where the $m\times(m-r)$ matrix $\mathbf{P}_{j,2}$, $j=1,2$, denotes the matrix of all $(m-r)$ orthogonal eigenvectors associated with the zero eigenvalue of $\mathbf{A}_j$. Unfortunately, many important hyperbolic systems such as the acoustic problem and Maxwell's equations do not satisfy these assumptions. In this manuscript, we show that the results in [10] hold for arbitrary linear symmetric hyperbolic systems with constant coefficient matrices $\mathbf{A}_1,\dots,\mathbf{A}_d$ in d space dimensions.

Thus, our *a posteriori* error estimates are implicit as they involve the solution of local problems for the error and are asymptotically exact provided the true solution is smooth enough. However solutions of hyperbolic systems may be discontinuous and thus our local theory does not hold on elements containing discontinuities or other singularities. Extensive computations suggest that the effectivity indices converge to unity under suitable adaptive mesh refinement even in the presence of discontinuities. Furthermore, our estimators have the following properties: (i) they yield accurate estimates of the true error in regions away from discontinuities, (ii) they underestimate the true error near singularities and in regions polluted by it, (iii) they may be used as error indicators for steering an adaptive mesh refinement algorithm. In some preliminary computations [5] we observed that an adaptive refinement strategy using an explicit error estimators for steering the adaptive refinement while using our error estimator for stopping the adaptive process can lead to much more efficient and robust algorithm than just using one estimator for both steering and stopping the adaptive process.

When used with a suitable adaptive algorithm our estimator tend to be asymptotically correct under adaptive mesh refinement in the presence of discontinuities. A possible explanation of this behavior is that the elements at the discontinuity, which are the source of high discretization and pollution errors, are refined such that their discretization and pollution errors are reduced to a harmless level. We also note that our error analysis does not include the effect of flux limiters and stabilization usually needed for high-order DG methods applied to hyperbolic systems with discontinuous solutions. This as well as the use of different numerical fluxes is currently under investigation.

This manuscript is organized as follows, in Section 2 we recall several results and preliminary results. In Section 3 we perform a local error analysis to investigate the asymptotic behavior of the local discretization error. In Section 4 we present our error estimation procedures and in Section 5 we present numerical results for several hyperbolic systems. We conclude and discuss our results in Section 6.

2. Problem formulation

Let d be the space dimension, $\mathbf{x}=(x_1,\dots,x_d)^t$ the space variable defined on a domain $\Omega=(0,1)^d\in\mathbb{R}^d$, and t the time variable defined on $[0,T]$.

Let $\mathbf{u}:[0,T]\times\Omega\to\mathbb{R}^m$ be the true solution of the linear symmetric hyperbolic system

$$\frac{\partial\mathbf{u}}{\partial t}+\sum_{i=1}^{d}\mathbf{A}_i\frac{\partial\mathbf{u}}{\partial x_i}=\mathbf{g}(t,\mathbf{x}),\quad \mathbf{x}\in\Omega,\ 0<t<T,\tag{2.1a}$$

with symmetric real constant coefficient matrices $\mathbf{A}_i\in\mathbb{R}^{m\times m}$, $1\leqslant i\leqslant d$, and subject to initial and boundary conditions

$$\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}),\quad \mathbf{x}\in\Omega,\tag{2.1b}$$

$$\left(\sum_{i=1}^{d}(\nu_i\mathbf{A}_i)^-\right)\mathbf{u}(t,\mathbf{x})=\left(\sum_{i=1}^{d}(\nu_i\mathbf{A}_i)^-\right)\mathbf{u}_B(t,\mathbf{x}),\quad \mathbf{x}\in\partial\Omega,\ 0<t<T,\tag{2.1c}$$

where $\partial\Omega$ denotes the boundary of Ω and $\boldsymbol{\nu}$ denotes the unit outward normal on $\partial\Omega$. Since symmetric matrices are diagonalizable with real eigenvalues, we define $\mathbf{M}^\pm$ for a symmetric matrix $\mathbf{M}\in\mathbb{R}^{m\times m}$ by writing

$$\mathbf{M}=\mathbf{P}\operatorname{diag}(\lambda_1,\dots,\lambda_m)\mathbf{P}^t,\quad \lambda_1,\dots,\lambda_m\in\mathbb{R},\tag{2.2a}$$

$$\mathbf{M}^+=\mathbf{P}\operatorname{diag}\bigl(\max(\lambda_1,0),\dots,\max(\lambda_m,0)\bigr)\mathbf{P}^t,\tag{2.2b}$$