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# A Dirichlet–Neumann reduced basis method for homogeneous domain decomposition problems

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ABSTRACT

Reduced basis methods allow efficient model reduction of parametrized partial differential equations. In the current paper, we consider a reduced basis method based on an iterative Dirichlet–Neumann coupling for homogeneous domain decomposition of elliptic PDE's. We gain very small basis sizes by an efficient treatment of problems with a-priori known geometry. Moreover iterative schemes may offer advantages over other approaches in the context of parallelization. We prove convergence of the iterative reduced scheme, derive rigorous a-posteriori error bounds and provide a full offline/online decomposition. Different methods for basis generation are investigated, in particular a variant of the POD-Greedy procedure. Experiments confirm the rigor of the error estimators and identify beneficial basis construction procedures.



## 1. Introduction

The reduced basis (RB) method has become a powerful approach for a fast and reliable treatment of parametrized partial differential equations (PDEs) in the last decades. Whenever solutions of such PDEs have to be approximated for plenty of parameters — the so-called multi-query context — or simulations have to be done in real-time, the runtime demands of precise approximation methods like Finite Elements (FE) or Finite Volumes is a big drawback. The RB method dissolves this problem by precomputing some snapshots of the solution manifold to generate a low dimensional approximation space. Subsequently, a reduced solution can be found rapidly in this low-dimensional space. The resulting reduced models can then be used in various settings, e.g. optimization with PDE-constraints [21,5], parameter studies or smartphone applications.

RB methods, which already emerged in the last century [1], have been studied for both steady, e.g. [23,26], and time dependent problems such as in [7,9]. In order to compute a basis for the low dimensional reduced space, one has to choose sample parameters. This is mostly done in an adaptive manner with the Greedy-algorithm [29]. For this procedure the a-posteriori error estimation is an essential tool, which can be very effective and often completely relies on known and rapidly computable quantities. A counterpart for time dependent problems is the so-called POD-Greedy method [9,20,25], which involves a proper orthogonal decomposition (POD) [14,30] of the projection error trajectories. Recently there has been much investigation in theoretical convergence analysis for the Greedy-algorithms [3,2,8].

Combining RB methods with a decomposition of the computational domain can be very advantageous. Functions living only on one subdomain can be used to build a global RB system. So solutions on the whole domain are no longer necessary. Furthermore the dimensionalities of the parameters associated to subproblems are often lower than in the monolithic approach. This technique led to the original RB element method (RBEM) [15–17]. The static condensation RBEM [11,6, 28] deals with reference components, which are connected via suitable ports. Especially, separate constructions for basis

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functions on interfaces and the domain interior are proposed. The RB hybrid method [13], being an extension of the RBEM, and the RDF method [12] also are combined methods recently developed in the context of fluidic networks. The RDF method is a new approach, where the basis of a high-dimensional FE discretization is used in variable-sized regions for the RB system. Domain decomposition techniques in general [24,27] are motivated by different purposes, such as parallelization of numerical simulations by assigning subdomains to different cores, or multiphysics, where different PDEs have to be solved on neighbouring domains. The resulting schemes are mostly of iterative nature.

In the current paper we develop an iterative RB method for linear elliptic problems based on initial results of [18]. We stick to the case where the geometry is known a-priori and use the Dirichlet–Neumann method, which is well known for the FE method [19]. In contrast to other approaches our RB scheme is exactly following the iterative nature of the detailed scheme, allowing new analytical statements as a basis for solutions to more complex problems. Especially, it is possible to develop an a-posteriori error estimator for discontinuous RB approximations. We propose suitable assumptions on the RB construction, which naturally arise from this context. In investigating different methods for basis generation, it is shown that these assumptions are realistic. To obtain computational efficiency we state a full offline/online decomposition of our procedure. The offline stage is parameter independent and involves all computations of high complexity. This allows for a very fast, parameter dependent online stage and is based on the parameter separability of the data.

The structure of the paper is as follows: In Section 2 the necessary notation and the problem formulation are specified. In Section 3 a full approximation scheme consisting of a detailed and a reduced procedure is presented. Results of analytical investigations follow in Section 4 and detailed considerations on computational aspects in Section 5. To complete the picture we present our numerical experiments in Section 6 and finally conclude with some closing remarks. In order to maintain readability of the main text, some proofs of analytical statements have been shifted to Appendix A.

## 2. Notation

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz-boundary $\partial\Omega$ and $x \in \overline{\Omega}$ the space variable. We introduce a Hilbert space $X \subset H_0^1(\Omega)$ with the norm $\|v\|_X := \|v\|_{H^1(\Omega)}$ which can be either finite or infinite dimensional. We now consider a decomposition of $\Omega$ into 2 subdomains, i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The interface $\Gamma$ is defined as $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. We assume that $\Omega_1$ and $\Omega_2$ have Lipschitz-boundaries and that $\Gamma$, $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$ have a nonvanishing $(n-1)$-dimensional measure. Several function spaces are defined according to the domain decomposition.

$$
\begin{aligned}
X_k &:= \{v|_{\Omega_k} \mid v \in X\},\\
X_k^0 &:= \{v \in X_k \mid \gamma v = 0\},\\
X_\Gamma &:= \gamma(X_1) = \gamma(X_2),
\end{aligned}
$$

where $k = 1, 2$. The operator $\gamma$ denotes the trace operator on $\Gamma$, where we do not notationally discriminate between the spaces $X_1$ or $X_2$, as it will always be clear from the context. It holds $X_1 \subset H^1(\Omega_1)$, $X_2 \subset H^1(\Omega_2)$ and $X_\Gamma \subset H_{00}^{1/2}(\Gamma)$. We equip the Hilbert spaces $X_k$, $k = 1, 2$, with the norms $\|v\|_{X_k} := \|v\|_{H^1(\Omega_k)}$ and $X_\Gamma$ with $\|g\|_{X_\Gamma} := \|g\|_{L_2(\Gamma)}$. Now let $\mathcal{P} \subset \mathbb{R}^P$, $P \in \mathbb{N}$, be the domain of the parameter $\mu \in \mathcal{P}$. We introduce the parametric elliptic variational problem for defining the parameter-dependent solution $u(\mu) \in X$:

$$
a\big(u(\mu), v; \mu\big) = f(v; \mu), \quad \forall v \in X, \tag{1}
$$

with a parametric, symmetric bilinear form $a : X \times X \times \mathcal{P} \to \mathbb{R}$ and a parametric linear form $f : X \times \mathcal{P} \to \mathbb{R}$. The function $u(\mu)$ is called *exact solution*. We assume that $a$ is continuous on $X$ for all $\mu \in \mathcal{P}$ with continuity constant

$$
M_X(\mu) := \sup_{v \in X\setminus\{0\}} \sup_{w \in X\setminus\{0\}} \frac{a(v, w; \mu)}{\|v\|_X \|w\|_X} < \infty
$$

and coercive on $X$ for all $\mu \in \mathcal{P}$ with coercivity constant

$$
\alpha_X(\mu) := \inf_{v \in X\setminus\{0\}} \frac{a(v, v; \mu)}{\|v\|_X^2} > 0.
$$

We also assume that $f$ is continuous and that $a$ and $f$ are parameter separable, i.e. there exist decompositions

$$
\begin{aligned}
a(v, w; \mu) &= \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(v, w), \quad \forall v, w \in X,\ \mu \in \mathcal{P},\\
f(v; \mu) &= \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(v), \quad \forall v \in X,\ \mu \in \mathcal{P},
\end{aligned}
$$