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A unified approach to identifying an unknown spacewise dependent source in a variable coefficient parabolic equation from final and integral overdeterminations

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ABSTRACT

An adjoint problem approach with subsequent conjugate gradient algorithm (CGA) for a class of problems of identification of an unknown spacewise dependent source in a variable coefficient parabolic equation $u_t = (k(x)u_x)_x + F(x)H(t)$, $(x,t) \in (0,l) \times (0,T]$ is proposed. The cases of final time and time-average, i.e. integral type, temperature observations are considered. We use well-known Tikhonov regularization method and show that the adjoint problems, corresponding to inverse problems ISPF1 and ISPF2 can uniquely be derived by the Lagrange multiplier method. This result allows us to obtain representation formula for the unique solutions of each regularized inverse problem. Using standard Fourier analysis, we show that series solutions for the case in which the governing parabolic equation has constant coefficient, coincide with the Picard's singular value decomposition. It is shown that use of these series solutions in CGA as an initial guess substantially reduces the number of iterations. A comparative numerical analysis between the proposed version of CGA and the Fourier method is performed using typical classes of sources, including oscillating and discontinuous functions. Numerical experiments for variable coefficient parabolic equation with different smoothness properties show the effectiveness of the proposed version of CGA.

1. Introduction

Heat source identification problems are the most commonly encountered inverse problems in heat conduction. These problems have been studied over several decades due to their significance not only in a variety of scientific and engineering applications, but also to their significance in the theory of inverse source problems for PDEs (see [1–6,8–11,13,15,17,16,18–21,28–34,37] and references therein). The final overdetermination for one-dimensional heat equation has first been considered by Tikhonov [35] in study of geophysical problems. In this work the heat equation with prescribed lateral and final data is studied in half-plane and the uniqueness of the bounded solution is proved. For parabolic equations in a bounded domain, when in addition to usual initial and boundary condition, a solution is given at the final time, well-posedness of inverse source problem has been proved by Isakov [19,20]. Certain existence, uniqueness and conditional stability questions for various inverse source problems have been analyzed in [3,8–11,28,31,34,38]. An adjoint problem approach for the inverse source problems related to linear and nonlinear parabolic equations has been proposed in [4–6]. This approach is then developed in [10,15,17], where, in particular, Fréchet differentiability of the cost functional and Lipschitz continuity of its gradient is proved. Some uniqueness theorems for inverse spacewise dependent source problems related to nonlinear parabolic and

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hyperbolic equations are given in [33]. Numerical algorithms for determination of a spacewise dependent source of the constant coefficient parabolic equation $u_t = ku_{xx} + F(x)$ are proposed in [13,16,21] (see also references therein). However these and other numerical algorithms can be implemented either when the thermal conductivity $k(x) > 0$ is a constant or the source term of a parabolic equation is a spacewise dependent only.

In this paper we study the inverse source problems of determining the unknown spacewise dependent source $F(x)$ in the parabolic problem

$$\begin{cases} u_t = \big(k(x)u_x\big)_x + F(x)H(t), & (x,t) \in \Omega_T := (0,l) \times (0,T]; \\ u(x,0) = u_0(x), & x \in (0,l); \\ \big(-k(x)u_x(x,t)\big)_{x=0} = g(t), \quad u(l,t) = 0, & t \in (0,T), \end{cases} \tag{1.1}$$

from the following types of observations. In the first inverse source problem, subsequently defined as ISPF1, the unknown spacewise dependent source $F(x)$ needs to be identified from the measured output data

$$u_T(x) := u(x,T;F), \quad x \in (0,l), \tag{1.2}$$

i.e. the supplementary temperature measurement $u_T(x)$ given at the final time $T > 0$.

In the second inverse source problem, subsequently defined as ISPF2, the unknown spacewise dependent source $F(x)$ needs to be identified from the integral of $u(x,t)$ over the time variable $t \in [0,T]$:

$$U_T(x) := \int_0^T u(x,t)\,dt, \quad x \in (0,l), \tag{1.3}$$

i.e. from the time-average temperature observation $U_T(x)$. Note that this kind of nonlocal or integral type observations arise, in particular, in mathematical modeling of settling mechanism, where $U_T(x)$ means an average density of sludge particles during the time $[0,T]$ (see [7] and references therein).

The source term $F(x)H(t)$ of the parabolic equation is assumed to be multiplicatives separable form, with the unknown spacewise dependent source $F(x)$ and known time-dependent heat source $H(t)$. Source terms of this separable form arise in various physical models. In particular, the equation $u_t = u_{xx} + F(x)H(t)$, with $H(t) = \exp(\kappa t)$, describes the heat process of radioisotope decay, with the decay rate $\kappa > 0$ [29]. These types of source terms appear also as a control term for the heat equation. Inverse source problems for separable form source terms parabolic and hyperbolic equations have first been studied in [11], then in [38,39].

We define the weak solution of the direct problem (1.1) as the function $u \in \mathring{V}^{1,0}(\Omega_T)$, satisfying the following integral identity [23]:

$$\iint_{\Omega_T} (-uv_t + ku_xv_x)\,dx\,dt = \iint_{\Omega_T} F(x)H(t)v(x,t)\,dx\,dt + \int_0^T g(t)v(0,t)\,dt, \quad \forall v \in \mathring{H}^{1,1}(\Omega_T), \tag{1.4}$$

with $v(x,T) = 0$. Here $V^{1,0}(\Omega_T) := C([0,T]; L^2[0,l]) \cap L^2((0,T); H^1(0,l))$ is the Banach space of functions with the norm

$$\|u\|_{V^{1,0}(\Omega_T)} := \max_{t\in[0,T]} \|u\|_{L^2[0,l]} + \|u_x\|_{L^2(\Omega_T)},$$

and $\mathring{V}^{1,0}(\Omega_T) := \{v \in V^{1,0}(\Omega_T):\ v(l,t) = 0,\ \forall t \in (0,T]\}$.

Here $H^{1,1}(\Omega_T)$ is the Sobolev space of functions with the norm [23]

$$\|u\|_{H^{1,1}(\Omega_T)} := \left(\iint_{\Omega_T} \big[u^2 + u_x^2 + u_t^2\big]\,dx\,dt\right)^{1/2},$$

and $\mathring{H}^{1,1}(\Omega_T) := \{v \in H^{1,1}(\Omega_T):\ v(l,t) = 0,\ \forall t \in (0,T)\}$.

Under the conditions

$$\begin{cases} k(x) \in L_\infty(0,l), \quad k^* \geqslant k(x) \geqslant k_* > 0; \\ u_0 \in L^2[0,l], \quad g(t) \in L^2[0,T], \quad F \in L^2[0,l], \quad H \in L^2[0,T], \end{cases} \tag{1.5}$$

the weak solution in $\mathring{V}^{1,0}(\Omega_T)$ exists and satisfies the following a priori estimate [27] (see also [23, Ch. 3.2] and [10, Ch. 10.1.1]):

$$\max_{t\in[0,T]} \big\|u(\cdot,t)\big\|_{L^2[0,l]} + \|u_x\|_{L^2(\Omega_T)} \leqslant C_0\big[\|u_0\|_{L^2(0,l)} + \|g\|_{L^2(0,T)} + \|F\|_{L^2[0,l]}\|H\|_{L^2[0,T]}\big]. \tag{1.6}$$