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A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation

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ABSTRACT

In this paper, we consider an inverse source problem for a time-fractional diffusion equation with variable coefficients in a general bounded domain. That is to determine a space-dependent source term in the time-fractional diffusion equation from a noisy final data. Based on a series expression of the solution, we can transform the original inverse problem into a first kind integral equation. The uniqueness and a conditional stability for the space-dependent source term can be obtained. Further, we propose a modified quasi-boundary value regularization method to deal with the inverse source problem and obtain two kinds of convergence rates by using an a priori and an a posteriori regularization parameter choice rule, respectively. Numerical examples in one-dimensional and two-dimensional cases are provided to show the effectiveness of the proposed method.

1. Introduction

The fractional diffusion equations can be used to describe the anomalous diffusion phenomena instead of the classical diffusion procedure and have attracted wide attentions in recent years. The time fractional diffusion equation arises when replacing the standard time derivative with a time fractional derivative and can be used to describe superdiffusion and subdiffusion phenomena [2,21,34,38]. The direct problems, i.e., initial value problem and initial boundary value problems for the time fractional diffusion equation have been studied extensively in recent years, for instance, on maximum principle [17], on some uniqueness and existence results [16,33], on numerical solutions by finite element methods [11,13] and finite difference methods [14,23,35,46], on analytic solutions [19,20,41].

However, for some practical problems, the part of boundary data, or initial data, or diffusion coefficients, or source term may not be given and we want to find them by additional measurement data which will yield to some fractional diffusion inverse problems. The early papers on inverse problems were provided by Murio in [22,24,25] for solving the sideways fractional heat equations by mollification methods. After that, some works on fractional inverse problems have been published. In [3], Cheng et al. considered an inverse problem for determining the order of fractional derivative and diffusion coefficient in a fractional diffusion equation and gave a uniqueness result. In [15], Liu et al. solved a backward problem for the time-fractional diffusion equation by a quasi-reversibility regularization method. Zheng et al. in [44,45] solved the Cauchy problems for the time fractional diffusion equations on a strip domain by a Fourier truncation method and a convolution regularization method. Qian in [31] used a modified kernel method to deal with a sideways fractional equation inverse problem. In [4,26,33,40,42], some inverse source problems were investigated. Furthermore, the nonlinear

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fractional inverse problems have been considered recently in [12,32]. To our knowledge, the research for inverse problems of fractional differential equations is a new topic and do not have too many results now.

In this paper, we consider the following inverse source problem for a time fractional diffusion equation with variable coefficients in a general bounded domain.

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth boundary $\partial\Omega$. The inverse source problem we considered for the time-fractional diffusion problem is to determine an unknown source term $f(x)$ from the following equations

$$\begin{cases} D_t^\alpha u(x,t) = (Lu)(x,t) + f(x)q(t), & x \in \Omega,\ t \in (0,T),\ 0<\alpha<1, & \text{(a)} \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), & \text{(b)} \\ u(x,0) = 0, & x \in \Omega, & \text{(c)} \\ u(x,T) = g(x), & x \in \overline{\Omega}, & \text{(d)} \end{cases} \tag{1.1}$$

where D_t^α is the Caputo fractional derivative of order α $(0<\alpha\leqslant 1)$ defined by

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}\, d\tau, & 0<\alpha<1, \\ u_t(x,t), & \alpha = 1; \end{cases} \tag{1.2}$$

and $-L$ is a symmetric uniformly elliptic operator defined on $D(-L) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Lu(x,t) = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}(x) \frac{\partial}{\partial x_j} u(x,t) \right) + c(x)u(x,t), \quad x \in \Omega, \tag{1.3}$$

in which the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \leqslant i,j \leqslant d,\ a_{ij} \in C^1(\overline{\Omega}), \tag{1.4}$$

$$\nu \sum_{i=1}^d \xi_i^2 \leqslant \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d,\ \text{for a constant } \nu > 0, \tag{1.5}$$

$$c(x) \leqslant 0, \quad x \in \overline{\Omega},\ c(x) \in C(\overline{\Omega}). \tag{1.6}$$

We assume the time-dependent source term $q(t)$ is given. The space-dependent source term $f(x)$ is determined from a noisy final data $g^\delta(x)$ which satisfies

$$\left\| g^\delta(x) - g(x) \right\| \leqslant \delta, \tag{1.7}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm and $\delta > 0$ is a noise level.

If $\alpha = 1$, the inverse source problem (1.1a)–(1.1d) is a classical ill-posed problem and has been studied in [7,36]. However for the fractional inverse source problem, to our knowledge, there are very few works, for example, Sakamoto et al. in [33] used the data $u(x_0,t)(x_0 \in \Omega)$ to determine $q(t)$ when $f(x)$ is given where the authors obtained a Lipschitz stability for $q(t)$. Zhang et al. in [43] used the Fourier truncation method and Wang et al. in [39] used the Tikhonov regularization to solve an inverse source problem with $q(t) \equiv 1$ in (1.1a) for one-dimensional case with special coefficients. In this study, we focus on a multi-dimensional problem with variable coefficients in a general bounded domain.

The quasi-boundary value method, also called nonlocal boundary value method in [9], is a regularization technique by replacing the final condition or boundary condition by a new approximate condition. This method has been used to solve some inverse problems for parabolic equation [1,5,9,10], hype-parabolic equations [37] and elliptic equations [6,8].

The standard quasi-boundary value method to deal with the inverse source problem (1.1a)–(1.1d) is to modify the final condition (1.1d) to form an approximate nonlocal problem

$$\begin{cases} D_t^\alpha v(x,t) = (Lv)(x,t) + f(x)q(t), & x \in \Omega,\ t \in (0,T), & \text{(a)} \\ v(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), & \text{(b)} \\ v(x,0) = 0, & x \in \Omega, & \text{(c)} \\ v(x,T) = g^\delta(x) - \mu f(x), & x \in \overline{\Omega}, & \text{(a)} \end{cases} \tag{1.8}$$

where μ plays a role of regularization parameter. It can be proved by the similar method in Section 4 that the best convergence rate for the source term $f(x)$ is $O(\delta^{1/2})$ under an a priori choice of regularization parameter and an a priori bound assumption to the exact $f(x)$ which is not best and can be improved.

In this study, we propose a modified version of quasi-boundary value method to solve the inverse source problem (1.1a)–(1.1d), i.e. replacing the final condition (1.1d) with a perturbed condition containing the value $(Lf)(x)$ as follows

$$\begin{cases} D_t^\alpha v(x,t) = (Lv)(x,t) + f(x)q(t), & x \in \Omega, t \in (0,T), & \text{(a)} \\ v(x,t) = 0, & x \in \partial\Omega, t \in (0,T), & \text{(b)} \\ v(x,0) = 0, & x \in \Omega, & \text{(c)} \\ v(x,T) = g^\delta(x) + \mu(Lf)(x), & x \in \overline{\Omega}. & \text{(d)} \end{cases} \tag{1.9}$$