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Convergence of a semi-discrete scheme for an abstract nonlinear second order evolution equation

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ABSTRACT

In the paper there is considered the Cauchy problem for an abstract nonlinear second order evolution equation in the Hilbert space. This equation represents a generalization of a nonlinear Kirchhoff-type beam equation. For approximate solution of this problem, we introduce a three-layer semi-discrete scheme, where the value of the gradient in the nonlinear term is taken at the middle point. This makes possible to reduce the finding of the approximate solution on each time step to solution of the linear problem. It is proved that the solution of the nonlinear discrete problem, as well as its corresponding difference analog of the first order derivative, is uniformly bounded. For the corresponding linear discrete problem, the high order *a priori* estimates are obtained using classic Chebyshev polynomials. Based on these facts, for nonlinear discrete problem, the *a priori* estimates are proved, whence the stability and error estimates of the approximate solution follow. Using the constructed scheme, numerical calculations for various test problems are carried out.

0. Introduction

In the present work, we consider the following abstract nonlinear evolution equation in a Hilbert space:

$$\frac{d^2u(t)}{dt^2} + A^2u(t) + a\left(\left\|A^{1/2}u\right\|^2\right)Au(t) + M\left(u(t)\right) = f(t),$$

where A is a self-adjoint, positive definite operator, $M(\cdot)$ is a Lipschitz continuous operator and $a(s)$ is a linear function. This equation (without Lipschitz continuous operator) represents a generalization of a nonlinear Kirchhoff-type beam equation (it was obtained by S. Woinowsky-Krieger, see [31]). Our aim is to find an approximate solution of Cauchy problem for this equation. For this purpose, we introduce a three-layer symmetric semi-discrete scheme, where the value of the gradient in the nonlinear term of the equation is taken at the middle point. This nuance is important, since for calculation of approximate solution on each time step, it is sufficient to inverse the linear operator.

Existence and uniqueness issues for local as well as global solutions of initial–boundary problem for a Kirchhoff string equation were first studied by Bernstein in 1940 (see [7]). The issues of solvability of the classical and generalized Kirchhoff equations were later considered by many authors: A. Arosio, S. Panizzi [1], L. Berselli, R. Manfrin [8], P. D'Ancona, S. Spagnolo [14,13], R. Manfrin [18], L.A. Medeiros [20], M. Matos [19], K. Nishihara [21], S. Panizzi [22]. In the works [1,14,13,18] and [21] issues of well-posedness and global solvability are thoroughly studied for a generalized Kirchhoff equation. In [22] the existence of global solution with low regularity is studied for Kirchhoff-type equations.

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An abstract analog of Kirchhoff-type beam equation is considered in the work by L.A. Medeiros [20], where the existence and uniqueness theorem for the regular solution of Cauchy problem is proved. The same abstract nonlinear equation, strengthened by first derivative with respect to time, is discussed in the works by P. Biler and E.H. Brito (see [9,10]), where most attention is paid to study of the behavior of Cauchy problem. We should note that a participation of the square of the main operator in the linear part of this equation essentially helps to obtain the necessary *a priori* estimates.

The following works are dedicated to approximate solutions of initial–boundary value problems for classical and generalized Kirchhoff equations: A.I. Christie, J. Sanz-Serna [12], I.S. Liu, M.A. Rincon [17], J. Peradze [23], J. Rogava, M. Tsiklauri [28]. In [23], the algorithm for approximate solution of classical Kirchhoff equation is thoroughly studied. This algorithm represents a combination of Galerkin method by spatial coordinate and finite difference time domain method.

J.M. Ball generalized the Kirchhoff beam equation by introducing damping terms, in order to account for the effect of external and internal damping (see [6]). For approximate solution of the initial–boundary value problem for this equation, the authors S.M. Choo, S.K. Chung in [11] suggested the finite difference method and studied the stability and convergence issues of the corresponding discrete problem.

As far as we know, approximate solving of abstract analog of the Kirchhoff-type beam equation are less studied. In the present work, investigation of stability and convergence of the semi-discrete scheme constructed for this equation is based upon two facts: (a) $(u_k - u_{k-1})/\tau$ and Au_k are uniformly bounded (u_k is an approximate solution, and τ is a time step); (b) for solution of the corresponding linear discrete problem, the *a priori* estimate is valid, where in the left-hand side is the main operator A with the positive power s, and in the right-hand side – A^{s-1}. The above-mentioned makes possible to weaken the nonlinear term in the given nonlinear equation to such degree that, taking into account fact (a), we can use Gronwall's lemma. In addition, it is not necessary to impose any restriction on τ here.

The results of the numerical calculations of test problems are presented at the end of the work. Note that there is used a second order accuracy three-point difference scheme with respect to the spatial coordinate. On the basis of numerical experiments, the convergence rate of the scheme is practically ascertained and it is shown that the constructed scheme describes well the behavior of oscillating solution.

1. Statement of the problem and semi-discrete scheme

Let us consider the Cauchy problem for an abstract equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + A^2u(t) + a\left(\left\|A^{1/2}u\right\|^2\right)Au(t) + M\big(u(t)\big) = f(t), \quad t \in [0, T], \tag{1.1}$$

$$u(0) = \varphi_0, \qquad \frac{du(0)}{dt} = \varphi_1, \tag{1.2}$$

where A is a self-adjoint (A does not depend on t), positive definite (generally unbounded) operator with the domain $D(A)$, which is everywhere dense in H, i.e. $\overline{D(A)} = H$, $A = A^*$ and

$$(Au, u) \geqslant \nu\|u\|^2, \quad \forall u \in D(A),\ \nu = const > 0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined the norm and scalar product in H, respectively; scalar function $a(s) = \lambda + s$, $\lambda > 0$; nonlinear operator $M(\cdot)$ satisfies Lipschitz condition; φ_0 and φ_1 are the given vectors from H; $u(t)$ is a continuous, twice continuously differentiable, sought function with values in H and $f(t)$ is the given continuous function with values in H.

As in the linear case (see [16, Theorem 1.5, p. 301]), the vector function $u(t)$ with values in H, defined on the interval $[0, T]$ is called a solution of the problem (1.1), (1.2) if it satisfies the following conditions: (a) $u(t)$ is twice continuously differentiable on the interval $[0, T]$; (b) $u(t) \in D(A^2)$ for any t from $[0, T]$ and the function $A^2u(t)$ is continuous; (c) $u(t)$ satisfies Eq. (1.1) on the $[0, T]$ interval and the initial condition (1.2). Here continuity and differentiability is meant by metric H. Existence and uniqueness of the solution of the problem (1.1), (1.2) (without Lipschitz continuous operator) is shown in [20].

Eq. (1.1) represents an abstract analog (without Lipschitz continuous operator) of Kirchhoff-type beam equation of the form (see [31]):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\lambda + \int_0^L u_\xi^2(\xi, t)\, d\xi\right)\frac{\partial^2 u}{\partial x^2} = f(x, t).$$

We seek a solution of the problem (1.1), (1.2) by the following semi-discrete scheme:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + A^2\frac{u_{k+1} + u_{k-1}}{2} + a\left(\left\|A^{1/2}u_k\right\|^2\right)\frac{Au_{k+1} + Au_{k-1}}{2} + M(u_k) = f_k, \tag{1.3}$$

where $k = 1, \dots, n-1$, $\tau = T/n$ $(n > 1)$, $f_k = f(t_k)$, $t_k = k\tau$, $u_0 = \varphi_0$.

As an approximate solution $u(t)$ of problem (1.1), (1.2) at point $t_k = k\tau$, we declare u_k, $u(t_k) \approx u_k$.

Regarding the scheme (1.3), we note that to calculate the approximate solution u_k $(k > 2)$ it is necessary to inverse the second order operator-polynomial with the positive coefficients. In case of factorization it is necessary to inverse operators $(I + \alpha_i A)$ $(i = 1, 2)$, where α_1 and α_2 are real and positive or complex conjugate numbers with positive real parts. Obviously in this case these operators are continuously invertible as A is a self-adjoint positive definite operator.