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Non-uniform multiresolution analysis for surfaces and applications

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ABSTRACT

Haar wavelet can exactly represent any piecewise constant function. Beam and Warming proved later that the supercompact wavelets can exactly represent any piecewise polynomial function in one variable, attaining higher level of accuracy by increasing the polynomial order of the supercompact wavelets. The approach of Beam and Warming, which is based on multiwavelets (family of wavelets) constructed in a one dimensional context, uses orthogonal basis defined over sequences of uniform partitions of $[0,1]$. The work of Beam and Warming has been recently extended by Fortes and Moncayo to the case of surfaces by using orthogonal basis defined over sequences of uniform triangulations of $[0,1]^2$. In that work the authors propose applications to data compression and to discontinuities detection, but both applications have the constraint that it is necessary to know information (at least) at the vertex of the triangulation, and so the data must be uniformly distributed. In the present work we overcome this constraint by considering a multiresolution scheme based on non-uniform triangulations. We develop the multiresolution algorithms and present two examples of the application of the algorithms to compress data and to detect discontinuities of data sets which need not to be uniformly distributed.

1. Introduction and preliminaries

It is a well-known fact that Haar wavelet can exactly represent any piecewise constant function. Beam and Warming generalized this result in [2] by proving that the supercompact wavelets can exactly represent any piecewise polynomial function in one variable, attaining higher level of accuracy by increasing the polynomial order of supercompact wavelet. The approach of Beam and Warming, developed in a context of uniform partitions of a nested grid hierarchy in the framework of Harten's multiscale representation (see e.g. [4,5]), is based on multiwavelets (family of wavelets) constructed in a 1D context.

In the literature we can find three extensions of [2]: In [1] the authors extend the results of [2] to sequence of non-uniform partitions of the unit interval. In [6] the initial approach [2] is extended to the case of multidimensional multiwavelets (3D). To this aim, the authors consider orthogonal basis defined as separable functions given by the product of three Legendre polynomials in one dimension. Later, in [3], the authors extend the theory developed in [2] to the case of surfaces by using non-separable orthogonal functions defined over sequences of uniform triangulations of $[0,1]^2$. In [3] the authors propose two applications: one to compress data and another one to detect discontinuities, but both applications have the constraint that all the vertex of the triangulations must be points in the data set, and since the considered triangulations are uniform, in order to apply the algorithms one needs to have (at least part of) the data set uniformly

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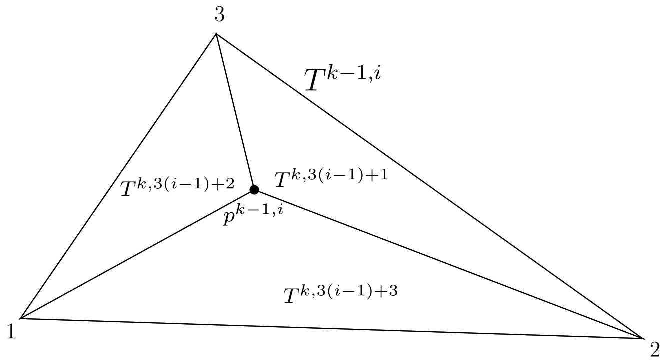

Fig. 1. Triangle $T^{k-1,i}$ of $\mathcal{T}^{k-1}$ and its subdivision into triangulation $\mathcal{T}^k$.

distributed, it being impossible to work with arbitrary data points. In the present work we extend the results of [3] to the non-uniform case, developing multiresolution algorithms to transfer information between non-uniform triangulations of $[0,1]^2$. This approach allows us to consider both applications of data compression and discontinuities detection in a more general frame: we still need to know information at the vertices of the triangulations, however since such triangulations need not be uniform, we can handle the problem of compress data and detect discontinuities in any set of data points. That is, we construct the triangulations depending on the data distributions. Of course, the higher the number of data to be handled, the more accurate both applications of data compression and discontinuity detection will be.

The paper is organized as follows: The next subsection is devoted to giving some notations and preliminaries. Section 2 presents a non-uniform multiresolution analysis (MRA) for $\mathbb{P}_1$-polynomial functions. We stress that although in [3] the authors developed the uniform MRA for general $\mathbb{P}_\ell$-polynomial, in this paper we opt to consider just $\mathbb{P}_1$-polynomial (anyway, the generalization to the non-uniform case for $\mathbb{P}_\ell$-polynomial is straightforward). The reason to consider just $\mathbb{P}_1$-polynomial is that the applications for data compression and discontinuities detection considered in this paper use only three data in each triangle (one for each of its vertices), and the functions in $\mathbb{P}_1$ are exactly the ones determined by three data. The corresponding decomposition and reconstruction procedures are explicitly stated. Section 3 deals with the definition of an MRA framework for more general functions. In Section 4 some applications are given. More precisely, the first application illustrates the high compressibility degree for the approximation of 3D-data at different resolution levels, while the second application shows the effectiveness of the given algorithms for the detection of discontinuities presented on a set of data. In a final Appendix A we show the details in obtaining the matrices $D^{k,m}$'s that appear in the multiresolution algorithms.

1.1. Sequence of triangulations

Let us consider the reference triangle T^* with vertices $(0,0)$, $(1,0)$ and $(0,1)$, and let $\{\varphi_j\}_{j=1}^3$ be an orthonormal basis of $\mathbb{P}_1(T^*)$.

Let $D \subset \mathbb{R}^2$ be a polygonal domain (an open polygonal non-empty connected set) and $\mathcal{T}^0 = \{T^{0,1}, \dots, T^{0,n_0}\}$ be a triangulation of $\overline{D}$ consisting of n_0 triangles. Let $\mathcal{D}^0$ be the set of knots of $\mathcal{T}^0$. For each $i = 1, \dots, n_0$, let $Af^{0,i}$ be the affinity that carries the triangle $T^{0,i}$ into T^*. Then, it is clear that $\{\varphi_j^{0,i}\}_{j=1}^3$, where

$$\varphi_j^{0,i} = \varphi_j \circ Af^{0,i}, \quad j = 1, 2, 3,$$

is an orthogonal basis of $\mathbb{P}_1(T^{0,i})$.

Now, we are going to define a sequence of triangulations $\{\mathcal{T}^k\}_{k \geqslant 1}$ inductively: Suppose that we have defined the triangulation $\mathcal{T}^{k-1} = \{T^{k-1,i}\}_{i=1}^{n_{k-1}}$, for $k \geqslant 1$, and that the three vertices of all triangles $T^{k-1,i}$ in $\mathcal{T}^{k-1}$ are numbered. Then we define the triangulation $\mathcal{T}^k$ as follows: For each $i = 1, \dots, n_{k-1}$, we consider an arbitrary point $p^{k-1,i} \in Int(T^{k-1,i})$ that will be used to decompose the triangle $T^{k-1,i}$ into three new subtriangles

$$\{T^{k,3(i-1)+1}, T^{k,3(i-1)+2}, T^{k,3(i-1)+3}\},$$

where $T^{k,3(i-1)+m}$ is the triangle with vertices $p^{k-1,i}$ and the vertices m' and m'' of $T^{k-1,i}$, with $m', m'' \neq m$ and $m' \neq m''$ (see Fig. 1).

We define

$$\mathcal{T}^k = \{T^{k,3(i-1)+m}\}_{\substack{i=1,\dots,n_{k-1} \\ m=1,2,3}}.$$

The triangulation $\mathcal{T}^k$ is finer than $\mathcal{T}^{k-1}$ and $card(\mathcal{T}^k) = n_k = 3n_{k-1}$. Moreover, if we denote $\mathcal{D}^k$ the set of all knots of $\mathcal{T}^k$, for $k \geqslant 1$, then it holds that

$$\mathcal{D}^k = \mathcal{D}^{k-1} \cup \{p^{k-1,i}\}_{i=1,\dots,n_{k-1}}.$$