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The extrapolation of numerical eigenvalues by finite elements for differential operators ☆

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ABSTRACT

This paper discusses the extrapolation of numerical eigenvalues by finite elements for differential operators and obtains the following new results: (a) By extending a theorem of eigenvalue error estimate, which was established by Osborn, a new expansion of eigenvalue error is obtained. Many achievements, which are about the asymptotic expansions of finite element methods of differential operator eigenvalue problems, are brought into the framework of functional analysis. (b) The Richardson extrapolation of nonconforming finite elements for multiple eigenvalues and splitting extrapolation of finite elements based on domain decomposition of non-selfadjoint differential operators for multiple eigenvalues are achieved. In addition, numerical examples are provided to support the theoretical analysis.

1. Introduction

The Richardson extrapolation is a well-known technique to construct high-order methods in numerical analysis. It is applicable to many problems, including ordinary or partial differential equations. All these applications are based on the existence of an error expansion for the discrete approximations in a single mesh parameter (see [16]). To produce more accurate approximations for partial differential equations, in the last 30 years, many scholars studied the Richardson extrapolation of finite element methods, e.g. see [1,2,4,9,10,13,24,25,28,33] and the references therein.

As for multidimensional problems, the Richardson extrapolation is costly since it considers just a single parameter. So, the splitting extrapolation, which is based on multivariate expansions with several mesh parameters, appears. Since 1980s, the splitting extrapolation has been developed widely in the numerical analysis community. The splitting extrapolation is a better technique to deal with the so-called curse of dimensionality and is also a highly parallel algorithm (see [20] and the book review [31]). It is especially important that the splitting extrapolation is also applied to the finite element methods, see [7,14,19,27,28], etc.

During the development of the extrapolation of finite element methods, the extrapolation for eigenvalue problems is an attractive issue, e.g. see [5,15,18,19,21,22,24,25,27,28,34]. Especially, [5] studied successfully the extrapolation of conforming finite elements for multiple eigenvalues of selfadjoint differential operator. However, to the best of our knowledge, there has no research on nonconforming finite elements extrapolations for multiple eigenvalues and the finite element extrapolations for multiple eigenvalues of non-selfadjoint differential operator. [22,24] discussed the extrapolation of nonconforming finite element eigenvalues, e.g., the asymptotic expansion of the EQ_1^{rot} element

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$$\lambda_h - \lambda = -\frac{h_1^2 + h_2^2}{3} \int_\Omega (\partial_1 \partial_2 u)^2 + \vartheta(h^4) \tag{1.1}$$

was proved where u is the limit of u_h, the eigenfunction corresponding to λ_h (see Theorem 3.1 in [22]). It is an important work. By using this expansion one can extrapolate the simple eigenvalue to obtain high-precision eigenvalues. However, in the case of multiple eigenvalues, from the spectral approximation theory we know that when h changes, the exact eigenfunction u which the finite element eigenfunction u_h approximates to also changes (e.g., see Theorem 7.4 in [3]), thus, u is related to h in the expansion. So it cannot be guaranteed that we use this expansion directly to extrapolate and get high-precision eigenvalues. As for the extrapolation of finite elements for non-selfadjoint differential operator eigenvalue problems, including the splitting extrapolation, [27] and [28] proved the asymptotic expansion for simple eigenvalues (see Theorem 2 in [27])

$$\lambda_h - \lambda = \sum_{i=1}^{l} \beta_i(u)\bar{h}_i^2 + \vartheta(h_0^4), \tag{1.2}$$

where $\beta_i(u)$ is independent of h. By this expansion the splitting extrapolation for simple eigenvalues can be achieved. But for multiple eigenvalues, $\beta_i(u)$ is related to h in (1.2). When the ascent of λ is larger than 1, not only is $\beta_i(u)$ related to h but also the accuracy of λ_h relates to the ascent which was pointed out in [28]. So we cannot use this expansion directly to get high-precision eigenvalues. The simple eigenvalue is a strong condition since the eigenvalue of non-selfadjoint problems is not simple in general and its ascent is probably larger than 1. This paper aims to study the extrapolation of finite elements for multiple eigenvalues including the case that the ascent is equal to or larger than 1.

We develop the previous corresponding investigations and obtain the main results which are in Sections 3 and 4 in this paper. Special works of this paper are as follows:

(a) In Section 3, we provide an eigenvalue error expansion (see (3.1)). This expansion is a simple extension of the estimate which was established by Osborn (see (2.4) in this paper). In many applications the first term on the right hand side of (3.1)/(2.4) is the dominant term and the second term is of higher order than the first one. The error is determined by the first term. Compared with (2.4), the advantage of (3.1) is that it indicates that the dominant term is effectively the size of the error. It is this feature that leads to the asymptotic formula for the error. Thus we bring the extrapolation of finite elements for differential operator eigenvalue problems into the framework of functional analysis.
(b) In Section 4, the asymptotic expansion of finite elements for differential operator eigenvalue problems is discussed by using Theorem 3.1. In Section 4.1 we give and prove the asymptotic expansions of EQ_1^{rot} element for multiple eigenvalues. In Section 4.2, for second-order non-selfadjoint differential operator eigenvalue problems, the splitting extrapolation based on domain decomposition is discussed. We throw off the assumption that λ is a simple eigenvalue in Theorem 2 in [27] and realize the splitting extrapolation of finite elements for multiple eigenvalues.

Besides, in Section 5, some numerical experiments are reported to support our theory.

In this paper, C denotes a positive constant independent of h, which may stand for different values at its different occurrences.

2. Preliminaries

Let X be a separable complex Banach space with norm $\|\cdot\|$ and conjugate pairs $\langle\cdot,\cdot\rangle$, respectively. In this section, let $T : X \to X$ be a nonzero compact linear operator, $T_h : X \to X$ and $\{T_h\}_{h>0}$ be a family of compact operators, and $\|T_h - T\| \to 0$ $(h \to 0)$. Consider the following eigenvalue problem:

$$Tu = \mu u, \tag{2.1}$$

and its approximation

$$T_h u_h = \mu_h u_h. \tag{2.2}$$

We use the eigenpairs of (2.2) to approximate to those of (2.1).

[12] has proved the following Lemma 2.1.

Lemma 2.1. *Let $\{\mu_j\}$ be an enumeration of the nonzero eigenvalues of T, each multiple according to its multiplicity. Then there exists an enumeration $\{\mu_{j,h}\}$ of the nonzero eigenvalues of T_h, with repetitions according to its multiplicity, such that*

$$\mu_{j,h} \to \mu_j \quad (h \to 0), \; j = 1, 2, \ldots. \tag{2.3}$$

Set $\lambda_j = \frac{1}{\mu_j}$, $\lambda_{j,h} = \frac{1}{\mu_{j,h}}$. In some papers μ_j and $\mu_{j,h}$ are called eigenvalues, and λ_j and $\lambda_{j,h}$ are called characteristic values. In our paper all of these are called eigenvalues.