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Finite element analysis of a projection-based stabilization method for the Darcy–Brinkman equations in double-diffusive convection

Aytekin Çıbık [a,*], Songül Kaya [b]

[a] Department of Mathematics, Gazi University, 06500, Ankara, Turkey
[b] Department of Mathematics, Middle East Technical University, 06800, Ankara, Turkey

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ABSTRACT

This paper presents a projection-based stabilization method of the double-diffusive convection in Darcy–Brinkman flow. In particular, it is concerned with the convergence analysis of the velocity, temperature and concentration in the time dependent case. Numerical experiments are presented to verify both the theory and the effectiveness of the method.

1. Introduction

Double-diffusion phenomena in a confined porous enclosure is modeled by the non-linear time dependent Darcy–Brinkman equations which read in dimensionless form

$$\begin{aligned}
&\partial_t \mathbf{u} - 2\nu\nabla\cdot\mathbb{D}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + Da^{-1}\mathbf{u} + \nabla p = (\beta_T T + \beta_C C)\mathbf{g} \quad \text{in } \left(0,t^*\right]\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0 \quad \text{in } \left(0,t^*\right]\times\Omega,\\
&\mathbf{u} = 0 \quad \text{on } \left(0,t^*\right]\times\partial\Omega,\\
&\partial_t T + \mathbf{u}\cdot\nabla T = \gamma\Delta T \quad \text{in } \left(0,t^*\right]\times\Omega,\\
&\partial_t C + \mathbf{u}\cdot\nabla C = D_c\Delta C \quad \text{in } \left(0,t^*\right]\times\Omega,\\
&T, C = 0 \quad \text{on } \Gamma_T, \qquad \frac{\partial T}{\partial\mathbf{n}}, \frac{\partial C}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma_B,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0, \qquad T(0,\mathbf{x}) = T_0, \qquad C(0,\mathbf{x}) = C_0 \quad \text{in } \Omega.
\end{aligned} \tag{1.1}$$

Here Ω be a regular bounded open set in $\mathbb{R}^d$ with $(d=2,3)$, $\partial\Omega = \Gamma_T\cup\Gamma_B$ is a polygonal or polyhedral boundary with $meas(\Gamma_T)>0$, and $\mathbf{u}$, p, T, C denote the velocity, pressure, temperature and concentration fields, respectively. The kinematic viscosity is $\nu>0$, the thermal diffusivity is $\gamma>0$, the mass diffusivity is $D_c>0$, the Darcy number is Da, the gravitational acceleration vector is $\mathbf{g}$, the velocity deformation tensor is $\mathbb{D}\mathbf{u} = (\nabla\mathbf{u}+\nabla\mathbf{u}^T)/2$ and the thermal and solutal expansion coefficients are β_T and β_C respectively. Some important dimensionless parameters which could be seen frequently in relevant

* Corresponding author.
E-mail addresses: abayram@gazi.edu.tr (A. Çıbık), smerdan@metu.edu.tr (S. Kaya).

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publications are the thermal Grashof number $Gr_T = \frac{\mathbf{g}\beta_T \triangle T \mathcal{H}^3}{\nu^2}$ and the solutal Grashof number $Gr_C = \frac{\mathbf{g}\beta_C \triangle C \mathcal{H}^3}{\nu^2}$, the buoyancy ratio $N = \frac{\beta_C \triangle C}{\beta_T \triangle T}$, the Prandtl number $Pr = \frac{\nu}{\gamma}$, the Schmidt number $Sc = \frac{\nu}{D_c}$, the Lewis number $Le = \frac{Sc}{Pr}$ and the Darcy number $Da = \frac{K}{\mathcal{H}^2}$ with given cavity height $\mathcal{H}$, a permeability K, $\triangle T$ and $\triangle C$ are the characteristics temperature and concentration differences along the enclosure, respectively.

The broad interest in solving Darcy–Brinkman (1.1) is caused by its wide variety of applications such as electrochemistry, metallurgy and geophysical systems, e.g., [30]. Problem (1.1) describes the double-diffusive convection phenomena which arises from the combined heat and mass transfer in porous medium. The system (1.1) models the physical phenomena in which the density gradients cause natural convection due to the coupled thermal and compositional effects. Both analytical and numerical treatments of the Darcy–Brinkman system are available in different flow configurations, see [8,30,31,36,20,6,22].

On the physical side, in the case of double-diffusive convection, the boundary layers are formed. The coupling of different boundary layers and circulating main core inside the enclosure are the fundamental difficulty in solving these systems analytically. One way to solve the Darcy–Brinkman system is to use finite element method. Unfortunately, it is well known that the standard finite element method generally yields inaccurate approximate solutions and performs poorly by displaying spurious oscillations along sharp layers [9,29]. A possible remedy is to use stabilization techniques to obtain physically correct numerical approximations [4,18].

One of the earliest studies regarding the projection-based stabilization was proposed in [14]. In this pioneering work, a two-scale decomposition of the flow field into large and small scale is proposed. The large scales are defined by a projection into appropriate spaces. While standard bubbles are used for the small scales, standard finite element spaces are used for the large scales. Based on the work of [14], different finite element discretizations of the projection-based stabilization method can be found, for instance in [21,12,16,19,17]. [24] proposes a method in which global stabilization is first added than these effects on the large scales of the solution are anti-diffused. Thus, the influence of stabilization is retained only on the resolved scales. In this paper, we use the framework of [24] to use a variationally consistent stabilization for the large scales of the Darcy–Brinkman equations to retain stabilization only on the small scales.

A wide range of papers concerning the experimental issues and continuous dependence and structural stability of solutions about the double-diffusive convection in a porous medium are available [32,26,27]. However, the topic of finite element error analysis on such kind of systems has not been considered yet. To the best of authors' knowledge, the error estimates of the finite element method with a projection-based stabilization idea applied to double-diffusive convection in porous media are not yet available.

The goal of the present paper is to extend the finite element error analysis of [3,28] for the natural convection equation to the double-diffusive convection case. We present a finite element error analysis of the projection-based stabilization for the Darcy–Brinkman system in the framework of [24,3]. We study convergence of the model in the semi-discrete case. Then, we present some numerical experiments to compare with the results of previous studies.

The plan of this paper is as follows. Section 2 presents the method for (1.1) to be studied and states known analytical results which will be used through the error analysis. Section 3 provides a stability estimation and a priori error analysis of the semi-discrete problem. Section 4 is devoted to present some numerical experiments in order to evaluate the performance of the method. Conclusion follows.

2. A projection-based finite element method

2.1. Variational formulation of a projection-based discretization scheme

For a bounded convex domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a polygonal or polyhedral boundary $\partial\Omega$, we use standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, with $p \in [1, \infty]$, $k \in \mathbb{R}$ [1]. The $L^p(\Omega)$ norms for $p \neq 2$ are explicitly denoted as $\| \cdot \|_{L^p}$. If $p = 2$, the inner product and the associated norm in $(L^2(\Omega))^d$ are denoted by $(\cdot,\cdot)$ and $\| \cdot \|$, respectively. Also, the norm in the Sobolev space $W^{k,2}(\Omega) = H^k(\Omega)$ is denoted with $\| \cdot \|_k$.

Let Y be a Banach space and for $0 < t < \infty$, the space $L^p(0, t; Y)$ consists of all functions defined on $(0, t) \times Y$ for which the norm

$$\|\mathbf{u}\|_{L^p(0,t;Y)} := \left(\int_0^t \|\mathbf{u}\|_Y^p \right)^{1/p}, \quad p \in [1, \infty),$$

is finite and with the usual modification in the definition of this space for $p = \infty$.

For vanishing boundary values, we define $H_0^1(\Omega)$ and its dual space, H^{-1}, its norm is defined by

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v} \in X} \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\nabla \mathbf{v}\|}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing. The symbol K stands for generic positive constant and may have different values at different places, but it does not depend on mesh sizes and other important parameters.