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Parallel pseudo-transient Newton–Krylov–Schwarz continuation algorithms for bifurcation analysis of incompressible sudden expansion flows

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ABSTRACT

We propose a parallel pseudo-transient continuation algorithm, in conjunction with a Newton–Krylov–Schwarz (NKS) algorithm, for the detection of the critical points of symmetry-breaking bifurcations in sudden expansion flows. One classical approach for examining the stability of a stationary solution to a system of ordinary differential equations (ODEs) is to apply the so-called a method-of-line approach, beginning with some perturbed stationary solution to a system of ODEs and then to investigate its time-dependent response. While the time accuracy is not our concern, the adaptability of time-step size is a key ingredient for the success of the algorithm in accelerating the time-marching process. To allow large time steps, unconditionally stable time integrators, such as the backward Euler's method, are often employed. As a result, the price paid is that at each time step, a large sparse nonlinear system of equations needs to be solved. The NKS is a good candidate solver for a system. Our numerical results obtained from a parallel machine show that our algorithm is robust and efficient and also verify, qualitatively, the bifurcation prediction with published results. Furthermore, imperfect pitchfork bifurcations are observed, especially for the case with a small expansion ratio, in which the occurrence of bifurcation points is delayed due to the stabilization terms in Galerkin/Least squares finite elements on asymmetric, unstructured meshes.

1. Introduction

In fluid dynamics, bifurcation phenomena, which provide the modes of transitions and instability when some physical parameter (such as the Reynolds number) is varied, are commonly observed. Some examples involve supercritical pitchfork (symmetry-breaking) bifurcations in laminar plane sudden expansion flows [1,3,6,12–14,22,34,41] and Hopf bifurcations in backward-facing step flows [18,38], in rotating cylindrical flows [32,44], and in lid-driven cavity flows [4,17,37,42]. One classical approach for examining the stability of a stationary solution is to simulate the discrete, time-dependent Navier–Stokes (NS) equations directly with some perturbations in the stationary solution and then to investigate whether the time-dependent response solution returns to the original solution or not after certain time steps. In conventional time-marching schemes, a method with a constant time step is commonly used to derive the numerical solution to reach a steady state. For example, Battaglia et al. [3], Drikakis [12], and Hawa and Rusak [22] applied this approach to simulate a time-dependent, symmetric sudden expansion channel flow. However, while time accuracy is not our concern, an algorithm with adaptive time step techniques seems to be more appropriate for our study because the solution at intermediate time

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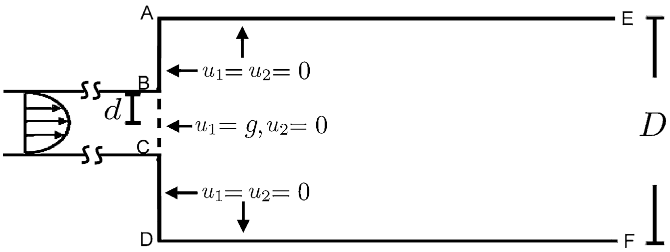

Fig. 1. The boundary conditions in the domain.

steps is not of interest, i.e., only the steady-state solution is required. In addition, parallel computing is necessary because such numerical simulation is a time-consuming task, even for 2D cases. For example, Kadja and Bergeles [27] reported that a typical test run in their numerical experiments (using a finite-volume fluid code with multigrid solvers on approximately 7000 grids) requires about five hours.

We propose the use of a parallel pseudo-transient continuation (Ψtc) algorithm in conjunction with Newton–Krylov–Schwarz (NKS) algorithms [5] to compute a stable symmetric/asymmetric solution usable for pitchfork bifurcation analysis, and we use the case of 2D sudden expansion flows as an example to study the performance of a parallel pseudo-transient Newton–Krylov–Schwarz continuation (ΨNKS) algorithm. For this purpose, the resulting time-singular system of ordinary differential equations (ODEs) is obtained by employing a stabilized finite element method for unsteady, incompressible NS equations as the spatial discretization. After employing unconditionally stable backward Euler's method as a time integrator, at each time step, the resulting nonlinear system is solved by a fully parallel NKS algorithm, where inexact Newton with backtracking as a nonlinear solver and an additive Schwarz preconditioned Krylov subspace-type method are used to solve the corresponding Jacobian systems.

Belonging to a family of continuation methods, the Ψtc algorithm [9,19,24,29] is one of the most popular globalization techniques for solving large, nonlinear algebraic systems of equations arising from the discretization of partial differential equations, with a broad range of applications in computational science and engineering, such as the 2D/3D Euler flow over a four-element airfoil [19,24], the incompressible Boussinesq flow in lid- and buoyancy-driven cavity, the reacting flow in a laminar diffusion flame [9], and the Poisson–Boltzmann equation [43]. The Φtc algorithm is particularly useful when a nonlinear iterative method, such as inexact Newton type method, fails to converge as the initial guess is far from the desired solution, or the desired solution has complicated characters but is not present in the initial iterate. The Ψtc algorithm first reformulates the original nonlinear system as a system of ODEs, then performs the numerical integration, starting from an initial guess, to obtain a steady-state solution by using an adaptive time-stepping technique. Kelley and coauthors [9,29] established the theoretical analysis for the global convergence and local convergence of the Ψtc algorithm: under certain assumptions, the Ψtc iteration exhibits a local superlinear convergence rate if unconstrained time steps are used, and the Ψtc iteration either converges globally to a desired solution or stagnates, provided that the initial time step selected is small enough.

The remainder of the paper is organized as follows. In the next section, we describe a system of ODEs arising in the sudden expansion flows and its symmetry-breaking bifurcations. In Section 3, we introduce the ΨNKS algorithms for the bifurcation analysis. In Section 4, we present some numerical results, including a study of the parallel performance of our algorithm and a prediction of the critical bifurcation points in distinct expansion ratios. Finally, the study's conclusions are presented.

2. Pitchfork bifurcation in sudden expansion flows

Consider the two-dimensional Newtonian viscous incompressible flow in a long channel of height $2d$ that suddenly expands symmetrically, at right angles to a channel of height D, where $D > 2d$. The expansion ratio (**ER**) is defined as the ratio of the channel height D to the upstream channel height $2d$. As shown in Fig. 1, only the downstream channel is included in a computational domain $\Omega \in R^2$ along with the boundary $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_{wall}$, where $\Gamma_{in} = \overline{BC}$, $\Gamma_{out} = \overline{EF}$, and $\Gamma_{wall} = \overline{AE} \cup \overline{AB} \cup \overline{CD} \cup \overline{DF}$.

The motion of such flows can be described by the unsteady incompressible NS equations written in a nondimensional form, as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \cdot \sigma = 0 & \text{in } \Omega \times (0, T), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma_{wall} \times (0, T), \\ u = g & \text{on } \Gamma_{in} \times (0, T), \\ \sigma \cdot n = 0 & \text{on } \Gamma_{out} \times (0, T), \end{cases} \tag{1}$$