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Bounds of the error of Gauss–Turán-type quadratures, II

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ABSTRACT

This paper is concerned with bounds on the remainder term of the Gauss–Turán quadrature formula,

$$R_{n,s}(f)=\int_{-1}^{1} f(t)w(t)\,dt-\sum_{\nu=1}^{n}\sum_{i=0}^{2s}\lambda_{i,\nu}f^{(i)}(\tau_\nu),$$

where

$$w(t)=w_{n,\ell}(t)=\left[U_{n-1}(t)/n\right]^{2\ell}\left(1-t^2\right)^{\ell-1/2}\quad(\ell\in\mathbb{N}),$$

U_{n-1} denotes the $(n-1)$th degree Chebyshev polynomial of the second kind and f is a function analytic in the interior of and continuous on the boundary of an ellipse with foci at ±1 and the sum of semi-axes $\varrho>1$.

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1. Introduction

Let w be an integrable (nonnegative) weight function on the interval $(-1,1)$, $n\in\mathbb{N}$ and $s\in\mathbb{N}_0$. It is well known that Gauss–Turán quadrature formula with multiple nodes,

$$\int_{-1}^{1} f(t)w(t)\,dt=\sum_{\nu=1}^{n}\sum_{i=0}^{2s}\lambda_{i,\nu}f^{(i)}(\tau_\nu)+R_{n,s}(f),\tag{1.1}$$

is exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The nodes τ_ν in (1.1) must be zeros of the corresponding s-orthogonal polynomials $\pi_n=\pi_{n,s}$ satisfying the following orthogonality conditions

$$\int_{-1}^{1}\pi_n(t)^{2s+1}t^k w(t)\,dt=0,\quad k=0,1,\dots,n-1.$$

Gauss–Turán quadrature formulae, or quadrature formulae with the highest degree of algebraic precision with multiple nodes, have extensively been studied in the last decades from both an algebraic and numerical point of view. Numerically

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stable methods for constructing nodes τ_ν and coefficients $\lambda_{i,\nu}$ can be found in [12] and [17]. Some interesting theoretical results concerning this theory have recently been obtained (see [16] (and references therein), [7,15]).

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation

$$R_{n,s}(f) = \frac{1}{2\pi i}\oint_\Gamma K_{n,s}(z) f(z)\,dz. \tag{1.2}$$

The kernel is given by

$$K_{n,s}(z;w) = \frac{\rho_{n,s}(z;w)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1,1], \tag{1.3}$$

where

$$\rho_{n,s}(z;w) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t} w(t)\,dt.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n,s}(\bar{z})| = |K_{n,s}(z)|$. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\bar{z})| = |K_{n,s}(z)|$ (see [8, Lemma 2.1]).

The integral representation (1.2) leads to a general error estimate, by using Hölder's inequality,

$$\bigl|R_{n,s}(f)\bigr| = \frac{1}{2\pi}\left|\oint_\Gamma K_{n,s}(z) f(z)\,dz\right| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma \bigl|K_{n,s}(z)\bigr|^r |dz|\right)^{1/r} \left(\oint_\Gamma \bigl|f(z)\bigr|^{r'} |dz|\right)^{1/r'}, \tag{1.4}$$

i.e.,

$$\bigl|R_{n,s}(f)\bigr| \leqslant \frac{1}{2\pi}\|K_{n,s}\|_r \|f\|_{r'}, \tag{1.5}$$

where $1 \leqslant r \leqslant +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} (\oint_\Gamma |f(z)|^r |dz|)^{1/r}, & 1 \leqslant r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ $(r' = 1)$ gives

$$\bigl|R_{n,s}(f)\bigr| \leqslant \frac{1}{2\pi}\Bigl(\max_{z\in\Gamma}\bigl|K_{n,s}(z)\bigr|\Bigr)\|f\|_1, \tag{1.6}$$

whereas for $r = 1$ $(r' = +\infty)$ we have

$$\bigl|R_{n,s}(f)\bigr| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma \bigl|K_{n,s}(z)\bigr|\,|dz|\right)\|f\|_\infty. \tag{1.7}$$

It is possible to obtain error bounds of the type (1.6) and (1.7) analytically (i.e., to calculate $\max_{z\in\Gamma}|K_{n,s}(z)|$ or $\oint_\Gamma |K_{n,s}(z)|\,|dz|$) only for weight functions which admit explicit Gauss–Turán quadrature formulae, i.e., in the cases when explicit formulae for corresponding s-orthogonal polynomials are known. There are only a couple of them.

In 1930, S. Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,dt \quad (k \geqslant 0).$$

This means that the Chebyshev polynomials T_n are s-orthogonal on $(-1,1)$ for each $s \geqslant 0$. Ossicini and Rosati [14] found three other weight functions $w_k(t)$ $(k = 2,3,4)$,

$$w_2(t) = \bigl(1-t^2\bigr)^{1/2+s}, \qquad w_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}, \qquad w_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}},$$

for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

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