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Stabilized finite element methods with anisotropic mesh refinement for the Oseen problem

Thomas Apel [a], Tobias Knopp [b], Gert Lube [c,*]

[a] *Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, D-85577 Neubiberg, Germany*
[b] *Institut für Aerodynamik und Strömungstechnik, Deutsches Zentrum für Luft- und Raumfahrt, Bunsenstr. 10, D-37073 Göttingen, Germany*
[c] *Institut für Numerische und Angewandte Mathematik, Georg-August-Universität Göttingen, Lotzestrasse 16-18, D-37083 Göttingen, Germany*

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Abstract

Nonstationary incompressible flow problems can be split into auxiliary problems of Oseen type. We present the analysis of conforming stabilized Galerkin methods of SUPG/PSPG-type with equal-order interpolation of velocity/pressure and with emphasis on structured anisotropic mesh refinement in boundary layers. We prove a modified inf-sup condition with a constant independent of the viscosity and of critical parameters of the mesh. Numerical tests support the results.

Keywords: Oseen equation; Channel flow; Stabilized finite element method; Anisotropic mesh

1. Introduction

We consider the nonstationary, incompressible Navier–Stokes problem

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

for velocity $\mathbf{u}$ and pressure p in a domain $\Omega \subset \mathbf{R}^d,\ d \leqslant 3$. In an outer loop, an A-stable low-order method (possibly with time step control) is applied. In an inner loop, we decouple and linearize the resulting system using a Newton-type iteration per time step. This leads to problems of Oseen type:

$$-\nu \Delta \mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega. \tag{4}$$

We consider stabilized conforming finite element schemes with equal-order interpolation of velocity/pressure for problem (3)–(4) with emphasis on anisotropic mesh refinement in boundary layers. The classical streamline upwind and pressure stabilization (SUPG/PSPG) techniques for the incompressible Navier–Stokes problem for equal-order in-

* Corresponding author.
E-mail addresses: thomas.apel@unibw.de (T. Apel), Tobias.Knopp@dlr.de (T. Knopp), lube@math.uni-goettingen.de (G. Lube).

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terpolation [4], together with additional stabilization of the divergence constraint (4), are well-understood on isotropic meshes [17,13].

Much less is known about the numerical analysis in the case of equal-order interpolation schemes with anisotropic mesh refinement for incompressible flow problems. The Stokes problem has been considered by Becker and Rannacher [3] and by Micheletti et al., e.g. in [14] for the Q1/Q1-case and the P1/P1-case, respectively. The extension to the Oseen problem seems to be new. Numerical experiments for the full Navier–Stokes problem, e.g. in [10,6], show the applicability of anisotropic mesh refinement for low-order schemes.

The paper is organized as follows: The stabilized finite element method for problem (3)–(4) is introduced in Section 2, where also existence and uniqueness of the discrete solution are proved. In Section 3 we focus on hybrid meshes with anisotropic mesh refinement of tensor product type in a boundary shear layer and a smooth transition to (in general unstructured) isotropic meshes away from the layer. We present a modified discrete inf-sup condition and a quasi-optimal a-priori estimate. Section 4 is devoted to error estimates and to the design of stabilization parameters. Some numerical results will be presented in Section 5.

Our investigation of anisotropic mesh refinement of shear layers is inspired by the work of Pieter Hemker who suggested a non-trivial model problem with shear layers in an advection–diffusion problem, see [9].

2. Stabilized finite element method for the Oseen problem

We consider the Oseen model:

$$L_{\mathrm{os}}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{7}$$

with $\mathbf{b}\in[H^1(\Omega)]^d$, $(\nabla\cdot\mathbf{b})(x)=0$, $\mathbf{f}\in[L^2(\Omega)]^d$ and constants $\nu>0$, $c\geqslant 0$. For brevity, we use homogeneous Dirichlet data; inhomogeneous data should be transferred to the right-hand side $\mathbf{f}$.

The variational formulation reads: find $U := \{\mathbf{u},p\}\in\mathbf{W} := \mathbf{V}\times\mathbf{Q} := [H_0^1(\Omega)]^d\times L_0^2(\Omega)$ with $L_0^2(\Omega) := \{q\in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$, s.t.

$$\mathcal{A}(\mathbf{b};U,V) = \mathcal{L}(V) \quad \forall V=\{\mathbf{v},q\}\in\mathbf{V}\times\mathbf{Q}, \tag{8}$$

with

$$\mathcal{A}(\mathbf{b};U,V) = (\nu\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + ((\mathbf{b}\cdot\nabla)\mathbf{u}+c\mathbf{u},\ \mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega + (q,\nabla\cdot\mathbf{u})_\Omega, \tag{9}$$

$$\mathcal{L}(V) = (\mathbf{f},\mathbf{v})_\Omega. \tag{10}$$

Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal/polyhedral domain Ω where each element $T\in\mathcal{T}_h$ is a smooth bijective image of a unit element $\hat{T}$, i.e., $T=F_T(\hat{T})$ for all $T\in\mathcal{T}_h$. Here, $\hat{T}$ is the unit simplex or the unit hypercube in $\mathbf{R}^d$ or, in the three-dimensional case, the unit triangular prism. A mixture of element types is admitted; in this case we use for each type the appropriate reference element. On this mesh, we consider Lagrangian finite elements of order $r\in\mathbf{N}$, i. e., $\mathcal{P}_r(\hat{T})$ denotes the polynomial space on the reference element that contains the set $\mathcal{P}_r$ of polynomials of degree r. We set

$$X_h^r = \{v\in C(\bar{\Omega}) \mid v|_T\circ F_T\in\mathcal{P}_r(\hat{T})\ \forall T\in\mathcal{T}_h\} \tag{11}$$

and introduce conforming equal-order finite element spaces for velocity and pressure

$$\mathbf{V}_h^r := [H_0^1(\Omega)\cap X_h^r]^d, \quad \mathbf{Q}_h^r := L_0^2(\Omega)\cap X_h^r, \quad r\in\mathbf{N}. \tag{12}$$

We use continuous pressure in order to avoid integrals on interelement boundaries after partial integration later on.

The Galerkin method reads: find $U=\{\mathbf{u},p\}\in\mathbf{W}_h^{r,r} := \mathbf{V}_h^r\times\mathbf{Q}_h^r$, s.t.

$$\mathcal{A}(\mathbf{b};U,V) = \mathcal{L}(V) \quad \forall V=\{\mathbf{v},q\}\in\mathbf{W}_h^{r,r}. \tag{13}$$

Well-known sources of instabilities of the Galerkin finite element method (13) stem from dominating advection and from the violation of the discrete Ladyzhenskaya–Babuška–Brezzi (LBB) condition for $\mathbf{V}_h^r\times\mathbf{Q}_h^r$. Moreover, in