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Interlacing theorems for the zeros of some orthogonal polynomials from different sequences

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ABSTRACT

We study the interlacing properties of the zeros of orthogonal polynomials p_n and r_m, $m = n$ or $n-1$ where $\{p_n\}_{n=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ are different sequences of orthogonal polynomials. The results obtained extend a conjecture by Askey, that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leqslant \alpha + 2$, showing that the conjecture is true not only for Jacobi polynomials but also holds for Meixner, Meixner–Pollaczek, Krawtchouk and Hahn polynomials with continuously shifted parameters. Numerical examples are given to illustrate cases where the zeros do not separate each other.

1. Introduction

It is well known that if $\{p_n\}_{n=1}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of p_n are real and simple and the zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ of p_n and $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ of p_{n-1} separate each other as follows (cf. [15, p. 61])

$$x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n} < x_{n-1,n-1} < x_{n,n}. \tag{1}$$

An important question that arises in the study of such interlacing properties is whether and when the zeros of two polynomials p_n and r_m, $m = n$ or $n-1$, where $\{p_n\}_{n=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$ are different sequences of orthogonal polynomials on the same interval, separate each other. In 1989, Askey [2, p. 28] conjectured that the zeros of Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\alpha+2,\beta)}$ interlace. He also posed the question whether the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\beta)}$ interlace when $\alpha < \gamma \leqslant \alpha + 2$.

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In [6] it was proved that the zeros of $p_n = P_n^{(\alpha,\beta)}$ and $r_n = P_n^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leqslant \alpha + 2$ and $\beta - 2 \leqslant \delta < \beta$. Furthermore, this paper also showed that the zeros of the $p_n = P_n^{(\alpha,\beta)}$ and $r_{n-1} = P_{n-1}^{(\gamma,\delta)}$ interlace when $\alpha < \gamma \leqslant \alpha + 2$ and $\beta < \delta \leqslant \beta + 2$. Results for the interlacing of the zeros of Gegenbauer polynomials, a special case of the Jacobi polynomials with $\alpha = \beta$, follow immediately but were proved separately in [5]. The zeros of Laguerre polynomials $p_n = L_n^{\alpha}$ and $r_m = L_m^{\gamma}$, where $m = n$ or $n - 1$ and $\alpha < \gamma \leqslant \alpha + 2$, also interlace (cf. [5]).

In this paper we aim to do a comprehensive study of the interlacing properties of the zeros of other classes of classical orthogonal polynomials, including discrete orthogonal polynomials such as Meixner, Krawtchouk and Hahn polynomials, where the parameters are shifted continuously.

The classical orthogonal polynomials of a discrete variable are used in various problems of theoretical and mathematical physics, group representation theory, for example the relationship between generalised spherical harmonics for SU(2) and Krawtchouk polynomials described in [10], as well as in representations of the three-dimensional rotation group (cf. [12]). An application of the Meixner–Pollaczek polynomials to quantum mechanics is discussed in [3].

Interlacing properties can easily be derived from the following simple result that has been proved with slight variations in different contexts, for example when considering quasi-orthogonality in [4, Theorem 3] or in dealing with polynomials associated with sequences of power moment functions [11, p. 117].

Lemma 1.1. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ and $n-1$ real zeros $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ respectively satisfying the interlacing property* (1) *on the* (*finite or infinite*) *interval* (c,d). *Assume that a polynomial f of degree n or $n-1$ satisfies the equation*

$$f(x) = a(x)p_n(x) + b(x)p_{n-1}(x). \tag{2}$$

If both a and b are continuous and have constant signs on (c,d), then all the zeros of f are real and simple, the zeros of f and p_n interlace and the zeros of f and p_{n-1} interlace.

In other words we have “triple interlacing" and there are several possibilities for the arrangement of the zeros. Let

$$t_1 < t_2 < \cdots < t_n \quad \text{be the zeros of } f.$$

Then, if f is of degree n, either

$$x_{i,n} < t_i < x_{i,n-1} \quad \text{for all } i = 1,\dots,n-1 \quad \text{and} \quad x_{n,n} < t_n \tag{3}$$

or

$$x_{i-1,n-1} < t_i < x_{i,n} \quad \text{for } i = 2,\dots,n \quad \text{and} \quad t_1 < x_{1,n}. \tag{4}$$

It is possible to obtain necessary and sufficient conditions to distinguish between cases (3) and (4). The signs of $a(x)$ and $b(x)$ will play a role in the distinction. For example, when a and b are constants (depending on n), one may apply a generalisation of [4, Theorem 3] or [8, Theorem 5].

Corollary 1.2. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ and $n-1$ real zeros $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ respectively satisfying the interlacing property* (1) *on the* (*finite or infinite*) *interval* (c,d) *and let f be a polynomial of degree n or $n-1$ satisfying* (2). *If b is continuous and has constant sign on (c,d), then all the zeros of f are real and simple and the zeros of f and p_n interlace.*

When f has degree n, there are two possibilities for the interlacing pattern. Let

$$t_1 < t_2 < \cdots < t_n \quad \text{be the zeros of } f$$

as before. Then either

$$x_{1,n} < t_1 < x_{2,n} < t_2 < \cdots < t_{n-1} < x_{n,n} < t_n \tag{5}$$

or

$$t_1 < x_{1,n} < t_2 < x_{2,n} < \cdots < t_{n-1} < x_{n-1,n} < t_n < x_{n,n} \tag{6}$$

and again it is possible to obtain necessary and sufficient conditions on $a(x)$ and $b(x)$ to distinguish between the two cases.

Corollary 1.3. *Let p_n and p_{n-1} be polynomials with n real zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ and $n-1$ real zeros $x_{1,n-1} < x_{2,n-1} < \cdots < x_{n-1,n-1}$ respectively satisfying the interlacing property* (1) *on the* (*finite or infinite*) *interval* (c,d) *and let f be a polynomial of degree $n-1$ satisfying* (2). *If a is continuous and has constant sign on (c,d), then all the zeros of f are real and simple and the zeros of f and p_{n-1} interlace.*