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Convergence analysis of the multiscale method for a class of convection–diffusion equations with highly oscillating coefficients ☆

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Abstract

This paper proposes a kind of multiscale method to solve the convection–diffusion type equation with highly oscillating coefficients, which arises in the studying of groundwater and solute transport in porous media. The introduced method is based on the framework of nonconforming finite element method, which can be considered as a realization of the heterogeneous multiscale method or variational multiscale method. The key point of the proposed method is to define a modified variational bilinear form with appropriate cell problems. Optimal estimate is proved for the error between the solution of the multiscale method and the homogenized solution under the assumption that the oscillating coefficients are periodic. While such a simplifying assumption is *not* required by our method, it allows us to use homogenization theory to obtain the asymptotic structure of the solution. Numerical experiments are carried out for the convection–diffusion type elliptic equations with periodic coefficients to demonstrate the accuracy of the proposed method. Moreover, we successfully use the method to solve the time dependent convection–diffusion equation which models the solute transport in a porous medium with a random log-normal relative permeability.

MSC: 65N30; 35B27; 65N12; 74Q05; 65F30

Keywords: Multiscale method; Homogenization; Convection–diffusion equation; Solute transport equation

1. Introduction

In this paper, we study the multiscale method for the following convection–diffusion equation

$$\begin{cases} -\nabla\cdot(\mathbf{b}^\varepsilon(x)u^\varepsilon(x)+\mathbf{a}^\varepsilon(x)\nabla u^\varepsilon(x)) = f(x), & x\in\Omega,\\ u^\varepsilon(x)=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded polyhedral domain with a Lipschitz boundary $\partial\Omega$, and $\varepsilon\ll 1$ is a parameter that represents the ratio of the smallest and largest scales in the problem. Problems of the type (1.1) are related to the studying of groundwater and solute transport in porous media (see [5,15]).

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In order to motivate the numerical method for problem (1.1), we recall some recent results for the typical problem

$$\begin{cases} -\nabla\cdot(\mathbf{a}^{\varepsilon}(x)\nabla u^{\varepsilon}(x)) = f(x), & x\in\Omega,\\ u^{\varepsilon}(x)=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $\mathbf{a}^{\varepsilon}(x) = (a^{\varepsilon}_{ij}(x))$ is a symmetric, positive definite, bounded tensor. Problem (1.2) can be used to describe several important practical problems such as flow in porous media and mechanical properties of composite materials. In general case, the coefficient $\mathbf{a}^{\varepsilon}(x)$ is a randomly varying function which may have a widely different length scales. It is difficult to solve this full fine scale problem via the traditional numerical methods which need to use spatial discretization that are capable of resolving all these length scales. To overcome this difficulty, many methods are designed to solve the problem on grids that are coarser than the scale of oscillations. The representative methods are the generalized finite element method [3,4] and the multiscale finite element method (MsFEM) [21,30,31], which solve the problem (1.2) on a coarse-scale mesh by modifying the basis functions in the framework of the finite element method. The modified basis functions contain the small scale information within each element. By solving the problem (1.2) in the modified basis function space, they get a good approximation to the full fine scale solution.

On the other hand, from an engineering perspective, macroscopic properties of the solutions are often of more importance. A well-known method is the homogenization technique by which the macro-solution can be obtained through solving the corresponding homogenized equation [6,34]. Homogenization theory and its application show that the homogenized solution can capture the macroscopic properties of the full fine scale solution effectively. It is obvious that we can resolve the homogenized equation by the standard finite element method on a coarse-scale mesh since the homogenized coefficient $\mathbf{a}^*$ is independent of ε (see [6]). Unfortunately, the analytical computational formula of $\mathbf{a}^*$ can only be obtained in some special cases. For the general case such as natural porous media, where $\mathbf{a}^{\varepsilon}$ has continuous scales, the analytical formula does not exist. Thus, various methods of upscaling or numerical homogenization have been developed, which approximate the original governing equations by another, often of the same form, with known coefficients that can be solved with fewer computing resources. The small-scale details of the medium are lumped into a few representative effective parameters on a coarse scale which preserve the large-scale behavior of the medium. Many works have been done based on similar ideas. Some of them form the coarse-scale equations with an explicit form whose effective parameters can be calculated by local problems (see [17,24,42] and references therein). In contrast, the other works do not express the coarse-scale equations explicitly. In their methods, the effective equations are carried out through the simulation implicitly and are generally formed and solved numerically. See, for instance, the wavelet homogenization techniques [16,22], the multigrid numerical homogenization techniques [26,36], the subgrid upscaling method [1,2], and the heterogeneous multiscale method (HMM) [18–20]. We refer the reader to [35] for an overview of the above methods.

However, to our knowledge, there are not so many works on how to solve Eq. (1.1) which has the strongly varying diffusion and convection coefficients simultaneously (see [29,38] for some works concerning the upscaling of convection–diffusion equations). In the literature, most of the multiscale methods concerning convection–diffusion problems are proposed to the convection-dominated case. See, for example, the variational multiscale method or residual-free bubbles method [8,10,28,33]. Recently, a new upscaling method was introduced in [11] to solve the solute transport problem which is a time dependent convection–diffusion equation. The basic idea of the method is to incorporate the upscaling procedure into the implementing of the finite element method, which can be seen as the implementation of the generalized finite element method. The most interesting part of the method is that the effective coefficients, both the diffusivity and velocity, are computed by using the solutions of the same local problem which is a second order elliptic equation that only uses the fine scale information of the diffusion coefficient.

Due to the complex heterogeneity of natural media, the convection and diffusion coefficients of the considered solute transport problem in [11] are highly oscillating. In this paper, we try to solve this kind of convection–diffusion equation in a different framework. Based on some analysis of the existed multiscale methods (see Section 2 below), we introduce a new numerical method which falls with the framework of the heterogeneous multiscale method. The main idea of the proposed method is to redefine the bilinear form of the finite element method by use of the solutions of the cell problems which are local elliptic problems. We emphasize that the definitions of the new bilinear form and the corresponding cell problems are not the direct development of the heterogeneous multiscale method for the elliptic problem (1.2). The unexpected difference is the definition of the cell problem which only uses the second order term information. We remark this idea has been used in the upscaling method proposed in [11]. Due to the simple definition of the cell problem, the proposed method costs much less than the full fine scale solver. Moreover, it is easy to see