Applied Numerical Mathematics 59 (2009) 1709–1719

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A fast numerical solution method for two dimensional Fredholm integral equations of the second kind ☆

Wen-Jing Xie, Fu-Rong Lin *

Department of Mathematics, Shantou University, Shantou Guangdong 515063, China

ARTICLE INFO

Article history:
Received 23 August 2006
Received in revised form 3 July 2008
Accepted 30 January 2009
Available online 7 February 2009

MSC:
45L10
65R20

Keywords:
Integral equation
Polynomial interpolation
Approximate matrix
Residual correction scheme

ABSTRACT

In this paper we consider numerical solution methods for two dimensional Fredholm integral equation of the second kind

$$f(x,y)-\int_{-1}^{1}\int_{-1}^{1}a(x,y,u,v)f(u,v)\,du\,dv=g(x,y),\quad (x,y)\in[-1,1]\times[-1,1],$$

where $a(x,y,u,v)$ is smooth and $g(x,y)$ is in $L^2[-1,1]^2$. We discuss polynomial interpolation methods for four-variable functions and then use the interpolating polynomial to approximate the kernel function $a(x,y,u,v)$. Based on the approximation we deduce fast matrix-vector multiplication algorithms and efficient preconditioners for the above two dimensional integral equations. The residual correction scheme is used to solve the discretization linear system.

1. Introduction

Many problems in engineering and mechanics can be transformed into two dimensional Fredholm integral equations of the second kind. For example, it is usually required to solve Fredholm integral equations in the calculation of plasma physics [11]. There are many works on developing and analyzing numerical methods for solving Fredholm integral equations of the second kind (see [1–4,6,10]).

In recent years, a number of algorithms for the fast numerical solution of integral equations of the second kind have been developed; see, e.g., [1,8,13,14,18–20]. The fast multipole method proposed in [13] combines the use of low-degree polynomial interpolation of the kernel functions with a divide-and-conquer strategy. For kernel functions that are Coulombic or gravitational in nature, it results in an order $O(N)$ algorithm for the matrix-vector multiplications, where N is the number of quadrature points used in the discretization. In [8], an $O(N\log N)$ algorithm was developed by exploiting the connections between the use of wavelets and their applications on Calderon–Zygmund operators. In [18], we proposed an approximation scheme to obtain low rank approximation of the discretization matrix. By using preconditioned iterative methods such as the residual correction (RC) scheme or the preconditioned conjugate gradient (PCG) method, the total cost for solving the integral equation with smooth kernel is $O(N)$ operations (ops).

The papers [14,19,20] are concerned with fast algorithms for two dimensional integral equations with weakly singular kernel functions. In [14] and [19], different versions of fast multipole methods are proposed to solve integral equations with weakly singular kernels. In [20], the authors developed a fast wavelet collocation method for the integral equations defined

☆ The research was supported by NSF of China No. 10271070.
* Corresponding author.
E-mail address: frlin@stu.edu.cn (F.-R. Lin).

0168-9274/$30.00 © 2009 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.apnum.2009.01.009

on polygons. Besides fast algorithms, there are many investigations on improving the accuracy of numerical solutions, see for instance [15] and [21]. In [15], a Nyström method for two dimensional Fredholm integral equation of second kind was used. The asymptotic expansion of numerical solution is obtained and then the Richardson extrapolation method is used to enhance the precision of the numerical solution. In [21], based on the finite-element solution, an iterative correction method was proposed. It shows that if the kernel function is smooth, the precision could be improved considerably.

In this paper, we consider the numerical solution of two dimensional Fredholm integral equation of the second kind with smooth kernel:

$$f(x,y) - \int_{-1}^{1}\int_{-1}^{1} a(x,y,u,v) f(u,v)\,du\,dv = g(x,y), \quad (x,y) \in [-1,1]\times[-1,1], \tag{1.1}$$

where the kernel function $a(x,y,u,v)$ is smooth in $[-1,1]^4$, the unknown function $f(x,y)$ and the right-hand side function $g(x,y)$ are in $L^2[-1,1]^2$. For the case where the integration domain is not $[-1,1]^2$, say $[\alpha,\beta]^2$, we can use the linear transformation

$$\begin{cases} x' = (2x-\alpha-\beta)/(\beta-\alpha), \\ y' = (2y-\alpha-\beta)/(\beta-\alpha) \end{cases}$$

to transform the domain into $[-1,1]^2$.

We discretize the equation by using certain numerical quadrature. Let $-1 \leqslant t_1^{(N)} < t_2^{(N)} < \cdots < t_N^{(N)} \leqslant 1$ be the quadrature points and $w_i^{(N)}$, $i = 1,2,\ldots,N$, be the corresponding weights, a numerical quadrature is defined as

$$\int_{-1}^{1} f(t)\,dt \approx \sum_{i=1}^{N} w_i^{(N)} f\big(t_i^{(N)}\big). \tag{1.2}$$

Typical examples include Gaussian rules, repeated Gaussian rules, and repeated Newton–Cotes rules, etc., see for instance [10]. Extending the above formula to two dimensional case, we get the following approximate system for (1.1):

$$f(x,y) - \sum_{i=1}^{N}\sum_{j=1}^{N} w_i^{(N)} w_j^{(N)} a\big(x,y,t_i^{(N)},t_j^{(N)}\big) f\big(t_i^{(N)},t_j^{(N)}\big) = g(x,y), \quad x,y \in [-1,1].$$

In particular, we get the discretization linear system of (1.1):

$$(I - A\tilde{W})\mathbf{f} = \mathbf{g}, \tag{1.3}$$

where $\mathbf{f}$ and $\mathbf{g}$ are vectors obtained by reordering the matrices $[f(t_i^{(N)},t_j^{(N)})]_{i,j=1}^N$ and $[g(t_i^{(N)},t_j^{(N)})]_{i,j=1}^N$ row-by-row respectively, I is the identity matrix, $\tilde{W} = W \otimes W$ with W the weight matrix $\operatorname{diag}(w_1^{(N)}, w_2^{(N)}, \ldots, w_N^{(N)})$, and

$$A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} & \cdots & A^{(1,N)} \\ A^{(2,1)} & A^{(2,2)} & \cdots & A^{(2,N)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(N,1)} & A^{(N,2)} & \cdots & A^{(N,N)} \end{pmatrix} \tag{1.4}$$

with $A^{(i,j)} = [a(t_i^{(N)},t_l^{(N)},t_j^{(N)},t_m^{(N)})]_{l,m=1}^N$. Here $\otimes$ denotes the Kronecker tensor product.

To solve the linear system (1.3) by classical direct methods such as Gaussian elimination method requires $O(N^6)$ operations (ops). For iterative methods such as the conjugate gradient (CG) method (see [12]), each iteration requires $O(N^4)$ ops. Therefore even for well-conditioned problems, the CG method requires $O(N^4)$ ops, which for large-scale problems is often prohibitive.

In this paper, we propose a fast numerical solution method for the linear system (1.3) by using the polynomial interpolation technique. That is, rather than solving the discretization linear system (1.3), we solve a Nyström approximation of the following equation

$$f^{[n]}(x,y) - \int_{-1}^{1}\int_{-1}^{1} a_n(x,y,u,v) f^{[n]}(u,v)\,du\,dv = g(x,y), \quad -1 \leqslant x,y \leqslant 1, \tag{1.5}$$

where $a_n(x,y,u,v)$ is an interpolation based approximation to $a(x,y,u,v)$ in which n Chebyshev nodes are used in each space variable. More precisely, we solve the following approximation equation of (1.3):

$$(I - A_a\tilde{W})\mathbf{f}^{[n]} = \mathbf{g}, \tag{1.6}$$