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Monotone iterative technique for numerical solutions of fourth-order nonlinear elliptic boundary value problems ☆

Yuan-Ming Wang [a,b]

[a] *Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China* [1]
[b] *Division of Computational Science, E-Institute of Shanghai Universities, Shanghai Normal University, Shanghai 200234, People's Republic of China*

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Abstract

This paper is concerned with finite difference solutions of a class of fourth-order nonlinear elliptic boundary value problems. The nonlinear function is not necessarily monotone. A new monotone iterative technique is developed, and three basic monotone iterative processes for the finite difference system are constructed. Several theoretical comparison results among the various monotone sequences are given. A simple and easily verified condition is obtained to guarantee a geometric convergence of the iterations. Numerical results for a model problem with known analytical solution are given.

Keywords: Fourth-order elliptic equations; Finite difference systems; Monotone iterations; Upper and lower solutions; Rate of convergence

1. Introduction

Boundary value problems of fourth-order differential equations have been given considerable attention in the literature, and most of the discussions are devoted to the existence, uniqueness, and multiplicity of solutions for the following two-point boundary value problem:

$$\begin{cases} u^{(iv)} = f(x, u, u''), & 0 < x < 1, \\ u(0) = u(1) = 0, & u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $f(x, u, u'')$ is, in general, a nonlinear function of u and u'' (cf. [1,2,7,11,12,16,18,19,28,32,36]). The above problem describes the static deflection of an elastic bending beam (with hinged ends) under a possible nonlinear loading (cf. [14,31]). It also describes the steady state of a prototype equation for phase transitions in condensed matter systems (cf. [13,33]), and is also useful in studying travelling waves in a suspension bridge (cf. [15,20]). In recent

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E-mail address: ymwang@math.ecnu.edu.cn (Y.-M. Wang).
[1] Corresponding address.

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years, attention has been given to the following fourth-order elliptic boundary value problem in a multidimensional domain and with the more general boundary condition:

$$\begin{cases} \Delta(k(x)\Delta u) = f(x, u, \Delta u), & x \in \Omega, \\ \mathcal{B}[u] = g(x), \quad \mathcal{B}[k\Delta u] = g^*(x), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded connected domain in $\mathbf{R}^n$ with boundary $\partial\Omega$, Δ is the Laplace operator, and

$$\mathcal{B}[w] \equiv \alpha_0 \partial w/\partial\nu + \beta_0(x)w$$

with $\partial/\partial\nu$ denoting the outward normal derivative on $\partial\Omega$ (cf. [9,20,21,23,24,27,30]). It is assumed that $f(x, u, v)$, $g(x)$, $g^*(x)$, and $\beta_0(x)$ are continuous functions in their respective domains, $k(x)$ is a strictly positive C^2-function on $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, and either $\alpha_0 = 0$, $\beta_0(x) \equiv 1$ (Dirichlet boundary condition) or $\alpha_0 = 1$, $\beta_0(x) \geqslant 0$ (Neumann or Robin boundary condition). A physical interpretation of (1.2) for the case $n = 2$ is that it governs the static deflection of a plate under a lateral loading. Here $k(x)$ is the stiffness of the plate, $g(x)$ and $g^*(x)$ are possible boundary sources, and $f(x, u, \Delta u)$ is the loading function, which may depend on the deflection and the curvature of the plate (cf. [31]).

The most discussions in the literature for (1.2) are again concerned with the existence, uniqueness, and multiplicity of solutions (cf. [9,20,21,23,30]). On the other hand, there are also a few papers that are devoted to the numerical methods for the computation of the solution but mostly for specific problems and linear equations (cf. [4,8,17]). For the general nonlinear problem (1.2), a finite difference-monotone iterative method is given in [24], where the problem (1.2) is discretized by the finite difference method, and three pointwise monotone iterative schemes are given to the corresponding nonlinear finite difference system. Another approach is given in [27], where two types of block monotone iterations, called block Jacobi and block Gauss–Seidel monotone iterations, are presented for the computation of solutions of the finite difference system. These block monotone iterations improve the rate of convergence of the pointwise monotone iterations given in [24] and can be easily computed by well-known computational algorithms for linear algebraic systems such as the Thomas algorithm (cf. [3,6]). However, the monotone convergence of the iterations in these works requires the monotone property of the nonlinear function $f(\cdot,u, v)$ in u. In this paper, we give a further investigation for the case where the nonlinear function $f(\cdot,u, v)$ is not necessarily monotone in u.

By formulating problem (1.2) as a coupled system of two second-order elliptic equations, we discretize the corresponding nonlinear equations into a system of nonlinear algebraic equations by the finite difference method. Our specific goal is to develop some pointwise monotone iterative schemes for the corresponding nonlinear finite difference system without any monotone requirement on the function $f(\cdot,u, v)$, including some comparisons and estimates for the rate of the convergence of the iterations. The removal of the monotone requirement on $f(\cdot,u, v)$ leads to a general computational algorithm for the numerical solutions of problem (1.2). Block Jacobi and block Gauss–Seidel monotone iterations for the nonmonotone function $f(\cdot,u, v)$ can be similarly developed.

The outline of the paper is as follows. In Section 2, we discretize the elliptic boundary value problem (1.2) into a coupled system of nonlinear finite difference equations. In Section 3, we develop a new monotone iterative technique for the computation of the finite difference solutions by the method of upper and lower solutions for nonmonotone function $f(\cdot,u, v)$. Three basic pointwise monotone iterative schemes are constructed and the monotone convergence of the iterations to a unique finite difference solution is proved. Section 4 is devoted to the rate of convergence of the iterations. We give several theoretical comparison results among the various monotone sequences, and obtain a simple and easily verified condition to guarantee a geometrically fast rate of convergence. In Section 5, we present some numerical results for a model problem with known analytical solution. These numerical results demonstrate theoretical analysis results and compare well with the known analytical solution. The final section is for some concluding remarks.

2. The finite difference system

To obtain a finite difference approximation for the boundary value problem (1.2) we let $v = -k\Delta u$ and transform problem (1.2) into the coupled system of the second-order elliptic equations:

$$\begin{cases} -\Delta u = v/k, \quad -\Delta v = f(x, u, -v/k), & x \in \Omega, \\ \mathcal{B}[u] = g^{(1)}(x), \quad \mathcal{B}[v] = g^{(2)}(x), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $g^{(1)}(x) = g(x)$ and $g^{(2)}(x) = -g^*(x)$. It is obvious that u is a solution of (1.2) if and only if (u, v) is a solution of (2.1).