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Convergence of an adaptive *hp* finite element strategy in one space dimension

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Abstract

We show uniform decrease in energy error for an *hp*-adaptive algorithm with automatic *hp* selection on the elliptic model boundary value problem. The result is based on a new marking strategy for the finite element refinement. In case of a solution with algebraic singularity we demonstrate that we achieve the known theoretical optimal error behavior.
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1. Introduction

The Finite Element Method is a flexible tool for the numerical solution of partial differential equations. One of the interesting features is the concept of a posteriori error estimation and adaptation of the finite element space to the solution [21,1]. The performance of the method can be improved either by mesh refinement (*h refinement*) or the use of higher order ansatz spaces (*p refinement*). Taking a combination of both methods (*hp refinement*) can lead to exponentially fast convergence with respect to the degrees of freedom [9–11], [19, Chapter 4.5]. This has been proved und numerically verified for several classes of problems. It is qualitatively clear where to perform h refinement and where p refinement and there are also a priori rules known for special cases [11]. However, one wants to find adaptive strategies for hp refinement that recovers the optimal exponential convergence behavior using only a posteriori information. Various such strategies have been suggested, e.g., [17,14,2,18,15,7,13]. The method proposed here is similar to the one used in [18], but we use a different marking procedure. Moreover, we can give an analysis that proves that this algorithm will lead to a uniform monotone decrease of the energy error in every step. This has not been proved before for an a posteriori strategy. Note that convergence of the algorithm does of course not imply optimal complexity of the algorithm. However, in a case of a solution with a singularity, the sequence of the numerically obtained errors follow the exponential law that is known to be the best one. Proving optimality is still an unsolved problem, but showing convergence is a first step in this direction.

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The main results of this paper are formulated in one space dimension. Many of the arguments can be generalized, but p uniform equivalence between exact and estimated error (as in Theorem 3) is not known to hold in higher space dimensions. Theoretical and numerical studies of two- and three-dimensional problems are subject of forthcoming research.

Notation. Let $\mathbb{P}_m$ for $m \in \mathbb{N}_0$ be the space of polynomials up to degree m. By $L^2(G)$, $H_0^1(G)$, and $H^m(G)$ we denote, for a domain $G \subset \mathbb{R}^d$, the Lebesgue and Sobolev spaces. The corresponding norms are $\|v\|^2_{L^2(G)} := \int_G |v|^2$, $\|v\|^2_{H_0^1(G)} := \|v'\|^2_{L^2(G)}$, and $\|v\|^2_{H^m(G)} := \sum_{s=0}^m \|v^{[s]}\|^2_{L^2(G)}$, respectively.

1.1. The model problem

Let $\Omega \subset \mathbb{R}$ be an open and bounded domain. Without loss of generality, we can assume that $\Omega := (0,1)$. For given functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}^d$ seek $u : \Omega \to \mathbb{R}$ with

$$\begin{aligned} -u'' &= f + g' \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The weak formulation. Multiplying (1) with test functions $v \in H_0^1(\Omega)$ and integrating by parts yields the problem: find $u \in H_0^1(\Omega)$ such that

$$\int_0^1 u'v' = \int_0^1 \{fv - gv'\} \quad \text{for all } v \in H_0^1(\Omega). \tag{2}$$

In this formulation, the assumption $f \in L^1(\Omega)$ and $g \in L^2(\Omega)$ leads to a well-posed problem. Note, that a discontinuity of the function g will induce a Dirac measure on the right-hand side of (1).

The Galerkin method. The idea of the Galerkin method consists in the approximation of $V := H_0^1(\Omega)$, $\|\cdot\|_V := \|\cdot\|_{H_0^1(\Omega)}$, by a finite dimensional space $V_N \subset V$ (with $\dim(V_N) = N \in \mathbb{N}$). The discrete problem is then to find $u \in V_N$ such that

$$\int_\Omega u_N' v_N' = \int_\Omega \{f v_N - g v_N'\} \quad \text{for all } v_N \in V_N. \tag{3}$$

Note, that we omit the influence of quadrature errors in the right-hand side. A unique solution to this problem exists and the error estimate is obtained by Cea's Theorem and interpolation estimates.

2. The finite element method

In the finite element method one constructs V_N as piecewise polynomial functions with respect to a decomposition of Ω.

2.1. Finite elements of varying polynomial order

We define the *grid* $\overline{\mathcal{G}}_n \subset \overline{\Omega}$ for $n \in \mathbb{N}$, to be a set of distinct *grid points*

$$\overline{\mathcal{G}}_n := \{x_0, \dots, x_{n+1}\colon\ 0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1\}.$$

Especially, the set of *interior grid points* is

$$\mathcal{G}_n := \overline{\mathcal{G}}_n \cap \Omega = \{x_1, \dots, x_n\}.$$

A *decomposition* of Ω is the set of intervals

$$\mathcal{K}_n := \bigl\{K = [x_{k-1}, x_k]\colon\ k \in \{1, \dots, n+1\}\bigr\}.$$

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