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Fourierization of the Legendre–Galerkin method and a new space–time spectral method

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Abstract

A set of Fourier-like basis functions is constructed for Legendre–Galerkin method for non-periodic boundary value problems and a new space–time spectral method is proposed. A complete error analysis is carried out for a linear parabolic equation and numerical results are presented for several typical linear and nonlinear equations.

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1. Introduction

Spectral method, in the context of numerical schemes, was introduced and popularized by Orszag's pioneer work in the early seventies. The term "spectral" was probably originated from the fact that the trigonometric functions $\{e^{ikx}\}$ are the eigenfunctions of the Laplace operator with periodic boundary conditions. This fact and the availability of Fast Fourier Transform (FFT) are two main advantages of the Fourier spectral method. Thus, using Fourier series to solve PDEs, with principal differential operator being the Laplace operator (or its power) with periodic boundary conditions, results in very attractive numerical algorithms. However, for problems with rigid boundaries, the eigenfunctions of Laplace operator (with non-periodic boundary conditions), although easily available in regular domains, are no longer good candidates as basis functions due to the Gibbs phenomenon (cf. [6]). In such cases, it is well known that one should use the eigenfunctions of the singular Sturm–Liouville operator, i.e., Jacobi polynomials with a suitable pair of indexes, e.g., Legendre and Chebyshev polynomials. Although these orthogonal polynomials have been successfully used for numerical approximation of PDEs (cf. [6,3,7,2]), there are still many situations where one wishes a set of Fourier-like basis functions would be available for non-periodic problems. The first objective of this paper is to construct such Fourier-like basis functions for elliptic boundary value problems. More precisely, we present an efficient and stable algorithm to construct basis functions which are mutually orthogonal with respect to both the

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L^2- and H^1-inner products. In particular, this set of basis functions is very convenient in developing efficient space–time spectral method which is the second topic of this paper.

Despite the fact that solutions of most time-dependent PDEs of current interest are much smoother in time than they are in space (except perhaps at $t = 0$ for parabolic-type equations), in most practical situations, high-order spectral methods in space are coupled with a low-order finite difference scheme in time, creating a mismatch in accuracy, and often resulting in a severe time step restriction which could be prohibitive for higher-order differential equations. Hence, it is plausible, for certain type of time-dependent PDEs, to use a spectral method for both space and time. We refer to [16] (see also [15,1] and the references therein) for an account on the rather limited activities in this area. It appears that the analysis for such space–time spectral methods are all restricted to periodic (in space) problems. Hence, the second objective of this paper is to present a new space–time spectral method based on a Legendre–Galerkin method in space and a dual-Petrov–Galerkin formulation in time. We also demonstrate that the use of Fourier-like basis functions in space may greatly simplify the implementation of the new space–time spectral method.

The rest of the paper is organized as follows. In the next section, we construct the Fourier-like basis functions. In Section 3, we propose the space–time spectral method and derive an optimal error estimate for a simple model problem. We present in Section 4 some illustrative numerical results. Some concluding remarks are given in the last section.

We now introduce some notations. Let ω be a positive weight function in a bounded domain Ω, and denote by $(u, v)_{\Omega,\omega} := \int_\Omega uv\omega \, d\Omega$ the inner product of $L^2_\omega(\Omega)$ whose norm is denoted by $\|\cdot\|_{\Omega,\omega}$. We use $H^m_\omega(\Omega)$ and $H^1_{0,\omega}(\Omega)$ to denote the usual weighted Sobolev spaces. For any Banach space X with norm $\|\cdot\|_X$, we define $L^2((a, b); X) = \{v\colon \int_a^b \|v\|_X^2 \, dt < +\infty\}$. In cases where no confusion would arise, ω (if $\omega \equiv 1$) and Ω may be dropped from the notations. We denote by P_M the space of all polynomials of degree $\leqslant M$, and by c a generic positive constant independent of any function and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leqslant cB$.

2. Fourier-like basis functions for the Legendre–Galerkin method

Many physical problems are governed by PDEs of the form

$$u_t = \mathcal{L}u + \mathcal{N}(u, t), \tag{2.1}$$

where $\mathcal{L}$ and $\mathcal{N}$ are higher-order linear and lower-order nonlinear operators, respectively. Typical examples include the Allen–Cahn, Burgers, Navier–Stokes, nonlinear Schrödinger, Cahn–Hilliard and Kuramoto–Sivashinsky equations. To approximate such problems, high-order stable numerical schemes in space and time are desirable but are not easy to construct due to the combinations of nonlinearities and stiffness.

A suitable semi-discretization in space of (2.1) leads to the following ODE system

$$\mathbf{M}\bar{u}_t = \mathbf{L}\bar{u} + \mathbf{N}(\bar{u}, t), \tag{2.2}$$

where $\bar{u}$ is an unknown vector consisting of either expansion coefficients (in terms of basis functions of the approximation space) of the approximate solution or the nodal values of the approximate solution, and $\mathbf{M}$ and $\mathbf{L}$ are the mass and stiffness matrices, respectively.

In a recent work [10], Kassam and Trefethen compared and evaluated five high-order time-stepping methods, i.e., implicit–explicit, split step, integrating factor, sliders and exponential time-differencing, coupled with spatial discretizations using Fourier and Chebyshev collocation methods for problem (2.1) with periodic and non-periodic boundary conditions. As pointed out in [10], having diagonal mass and stiffness matrices is crucial in the implementation and analysis of some time-stepping schemes. In particular, the *sliders* method (cf. [5]), in which different schemes are used for "fast", "medium" and "slow" modes, is designed essentially for periodic problems. However, for non-periodic problems, the commonly-used basis functions in a spectral approximation lead to either a diagonal mass matrix (as in the collocation method) or a diagonal stiffness matrix (as in the Legendre–Galerkin method [12]) but not both. Therefore, many of the high-order, efficient time stepping schemes cannot be directly designed, so it is of interest to construct Fourier-like basis functions for spatial discretization of non-periodic problems.

To demonstrate the main idea, we consider the following one-dimensional problem

$$\begin{aligned} &\partial_t u - \alpha \partial_x^2 u + \mathcal{N}(u, t) = 0, \quad x \in I := (-1, 1), \ t > 0, \\ &u(x, 0) = u_0(x), \quad x \in \bar{I}, \end{aligned} \tag{2.3}$$