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Finite difference approximations for some interface problems with variable coefficients

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Abstract

General elliptic interface problem with variable coefficients and curvilinear interface is transformed into analogous problem with rectilinear interface. For the numerical solution of transformed problem a finite difference scheme with averaged right-hand side is proposed. Convergence rate estimate in discrete W_2^1 norm, compatible with the smoothness of data, is obtained. Analogous parabolic problem with the singularity of "concentrated capacity" type is considered. Two finite difference schemes for its solution are proposed and investigated.

MSC: 65N06; 65M06; 65N15; 65M15

Keywords: Interface problem; Weak solution; Sobolev space; Finite differences

1. Introduction

Interface problems arise in different situations, for example, in heat conduction in composite materials or in problems with transmission conditions at the interfaces and singular sources. A survey of analytical results and models in physics and chemistry concerning the equations considered in this paper is given in [3,20,24,30,32]. Various forms of conjugation conditions satisfied by the solution and its derivatives on the interface are known [3,4,7,10,12,16,20,24]. In [16–20] convergence of finite difference method for different elliptic, parabolic and hyperbolic interface problems is studied. Elliptic and parabolic interface problems with non-zero jump in the flux across the smooth interface are considered in [4,10,28,29]. In the immersed interface method the jumps of the solution and its derivatives are utilized to modify the standard finite difference schemes in the neighborhood of the interface (see [23]). Curved interface problems within the framework of mortar finite element methods are analyzed in [8].

In the first part of the present work (Section 2) we investigate a general elliptic interface problem in rectangular domain, crossed by curvilinear interface. By suitable variable change the problem is transformed into analogous one with rectilinear interface. For the numerical solution of transformed problem a finite difference scheme with averaged right-hand side is proposed. Convergence rate estimate in discrete W_2^1 norm, compatible with the smoothness of

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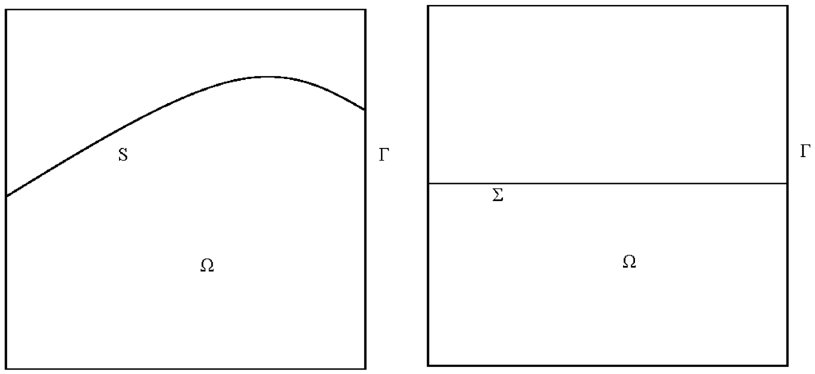

Fig. 1.

the data, is obtained. A part of these results was reported on the conference NAA'04-Rousse, Bulgaria, see [19]. Analogous problem for Laplace operator is considered in [12].

In the second part of the paper (Section 3) an analogous parabolic problem with line interface is considered. Two finite difference schemes for its numerical solution are proposed and investigated. For both schemes the convergence rate estimates in discrete Sobolev like norm $\widetilde{W}_2^{1,1/2}$ are obtained. The obtained convergence rates are compatible with the smoothness of the input data (up to a logarithmic factor of the mesh-size).

2. Elliptic problem

2.1. Problem with curvilinear interface

Let $\Omega = (0,1)^2$, $\Gamma = \partial\Omega$, and let S be a smooth curve intersecting Ω. For clarity, let S be defined by equation $\xi_2 = g(\xi_1)$, where $g \in \mathbb{C}^1[0,1]$ and $0 < g_0 \leqslant g(\xi_1) \leqslant g_1 < 1$, g_0, $g_1 = \text{const}$. In domain Ω we consider Dirichlet boundary value problem [19]

$$\mathcal{M}U + K(\xi)\delta_S(\xi)U = F(\xi) \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \Gamma, \tag{1}$$

where $\xi = (\xi_1, \xi_2)$,

$$\mathcal{M}U = -\sum_{i,j=1}^{2} A_{ij}(\xi)\frac{\partial^2 U}{\partial\xi_i\partial\xi_j} + 2\sum_{i=1}^{2} B_i(\xi)\frac{\partial U}{\partial\xi_i} + C(\xi)U, \quad A_{ij}(\xi) = A_{ji}(\xi)$$

is elliptic operator and $\delta_S(\xi)$ is Dirac distribution concentrated on S. The equality in (1) is treated in the sense of distributions.

The BVP (1) can be transformed into analogous one with the rectilinear interface. It can be easily verified that the change of variables $x = x(\xi)$, where

$$x_1 = \xi_1, \qquad x_2 = \frac{[1 - g(\xi_1)]\xi_2}{\xi_2 - 2\xi_2 g(\xi_1) + g(\xi_1)} \tag{2}$$

maps Ω onto Ω. The curve S is mapped onto straight line Σ: $x_2 = 1/2$ (see Fig. 1).

By change of variables (2) BVP (1) transforms to following one

$$\mathcal{L}u + k(x)\delta_\Sigma(x)u = f(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{3}$$

where $u(x) = U(\xi)$, $f(x) = F(\xi)$, $\delta_\Sigma(x) = \delta(x_2 - 1/2)$ is Dirac distribution concentrated on Σ, $k(x) = k(x_1) = K(x_1, g(x_1))\sqrt{1 + [g'(x_1)]^2}$, while coefficients of differential operator

$$\mathcal{L}u = -\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + 2\sum_{i=1}^{2} \hat{b}_i(x)\frac{\partial u}{\partial x_i} + \hat{c}(x)u, \quad a_{ij}(x) = a_{ji}(x)$$

can be expressed by coefficients of $\mathcal{M}$ and derivatives $\frac{\partial x_i}{\partial\xi_j}$, for example