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# High-order schemes for acoustic waveform simulation

Seongjai Kim [1], Hyeona Lim *

*Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762-5921, USA*

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**Abstract**

This article introduces a new fourth-order *implicit* time-stepping scheme for the numerical solution of the acoustic wave equation, as a variant of the conventional modified equation method. For an efficient simulation, the scheme incorporates a locally one-dimensional (LOD) procedure having the splitting error of $\mathcal{O}(\Delta t^4)$. Its stability and accuracy are compared with those of the standard explicit fourth-order scheme. It has been observed from various experiments for 2D problems that (a) the computational cost of the implicit LOD algorithm is only about 40% higher than that of the explicit method, for the problems of the same size, (b) the implicit LOD method produces less dispersive solutions in heterogeneous media, and (c) its numerical stability and accuracy match well those of the explicit method.


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*Keywords:* Acoustic wave; High-order method; Locally one-dimensional (LOD) method; Nonphysical dispersion

## 1. Introduction

Let $\Omega \subset \mathbb{R}^m$, $1 \leqslant m \leqslant 3$, be a bounded domain with its boundary $\Gamma = \partial\Omega$ and $J = (0, T]$ the time interval, $T > 0$. Consider the following acoustic wave equation:

$$\begin{aligned}
&\text{(a)}\quad \frac{1}{c^2}u_{tt} - \Delta u = S(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J,\\
&\text{(b)}\quad \frac{1}{c}u_t + u_\nu = 0, \quad (\mathbf{x}, t) \in \Gamma \times J,\\
&\text{(c)}\quad u(\mathbf{x}, 0) = g_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = g_1(\mathbf{x}), \quad \mathbf{x} \in \Omega,\ t = 0,
\end{aligned} \tag{1}$$

where $c = c(\mathbf{x}) > 0$ denotes the normal velocity of the wavefront, $S$ is the wave source/sink, $\nu$ denote the unit outer normal from $\Gamma$, and $g_0$ and $g_1$ are initial data.

Wave problems are often formulated in an unbounded domain. These problems can be solved numerically by first truncating the given unbounded domain, imposing a suitable absorbing boundary condition (ABC) on the (artificial) boundary of the truncated bounded domain, and then solving the resulting problem using discretization methods (e.g.,

* Corresponding author.
*E-mail addresses:* skim@math.msstate.edu (S. Kim), hlim@math.msstate.edu (H. Lim).
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finite differences, finite elements, and spectral methods). Eq. (1.b) has been popular as a simple-but-effective ABC, since introduced by Clayton and Engquist [3]. Eq. (1) has been extensively studied as a model problem for second-order hyperbolic problems by many authors; see, e.g. [1,2,5,10,11,13]. It is often the case that the source is given in the following form

$$S(\mathbf{x},t)=\delta(\mathbf{x}-\mathbf{x}_s)f(t),$$

where $\mathbf{x}_s\in\Omega$ is the source point. For the function $f$, the Ricker wavelet of frequency $\nu$ can be chosen, i.e.,

$$f(t)=\pi^2\nu^2\big(1-2\pi^2\nu^2t^2\big)e^{-\pi^2\nu^2t^2}.$$

In Geophysical applications, the wave equation (1) is often solved by explicit time-stepping schemes, which require to choose the time step size sufficiently small to satisfy the stability condition and to reduce numerical dispersion as well. Alternative conventional approaches for solving wave equations introduce an auxiliary variable to rewrite the equation as a first-order hyperbolic system. With these approaches one introduces new unknowns, which result in an increase in the number of variables in the discrete problems. Thus, there are good reasons to try to keep the formulation involving the second time-derivative and a scalar unknown. However, it has been known that with this formulation it is hard to construct methods combining good stability with high accuracy. In particular, it is hard to incorporate a high-order approximation of the ABC. In this paper we shall introduce a one-parameter family of three-level methods incorporating the locally one-dimensional (LOD) time-stepping procedure for an efficient simulation. It is analyzed to be unconditionally stable for the parameter in a certain range.

An outline of the article is as follows. In the next section, we first review the conventional methods: explicit (three-level) schemes and the two-level implicit scheme. Section 3 introduces a new three-level implicit scheme. A locally one-dimensional (LOD) perturbation having the splitting error in $\mathcal{O}(\Delta t^4)$ is considered for an efficient simulation. Its stability and computational complexity are compared with those of the standard explicit fourth-order scheme in the same section. In Section 4, we present some numerical results showing numerical stability, efficiency, and accuracy of the new scheme. In Section 5, we discuss strategies of incorporating high-order approximations of the ABC. The last section includes conclusions.

## 2. Preliminaries

In this section, we review conventional methods for the numerical solution of the wave equation (1). Let $\mathcal{A}$ denote an approximation of $-\Delta$ of order $p$, i.e.,

$$\mathcal{A}u\approx-\Delta u+\mathcal{O}\big(h^p\big),$$

where $h$ is the grid size; in most cases, $p$ is 2 or 4. Then, the semi-discrete equation for the acoustic wave equation reads

$$\frac{1}{c^2}v_{tt}+\mathcal{A}v=S. \tag{2}$$

(Here we have omitted the equations for the boundary and initial conditions, for a simpler presentation.)

It now remains to discretize the second-order system of ODEs (2) with respect to the time variable. Let $\Delta t$ be the time step size and $t^n=n\Delta t$. Define $v^n(\mathbf{x})=v(\mathbf{x},t^n)$. For a simpler presentation, we define the following difference operator

$$\bar\partial_{tt}v^n:=\frac{v^{n+1}-2v^n+v^{n-1}}{\Delta t^2}.$$

### 2.1. Explicit schemes

Explicit methods are still popular in the simulation of waveforms. We begin with the second-order scheme (in time) formulated as

$$\frac{1}{c^2}\bar\partial_{tt}v^n+\mathcal{A}v^n=S^n. \tag{3}$$