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Boundary element methods—An overview ☆

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Dedicated to Professor Dr. Wolfgang L. Wendland in Friendship and Admiration

Abstract

Variational methods for boundary integral equations deal with the weak formulations of boundary integral equations. Their numerical discretizations are known as the boundary element methods. This paper gives an overview of the method from both theoretical and numerical point of view. It summarizes the main results obtained by the author and his collaborators over the last 30 years. Fundamental theory and various applications will be illustrated through simple examples. Some numerical experiments in elasticity as well as in fluid mechanics will be included to demonstrate the efficiency of the methods.

MSC: 35J20; 45B05; 65N30

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1. Introduction

Variational methods for boundary integral equations deal with the weak formulations of boundary integral equations. Their numerical discretizations are generally known as the boundary element methods (BEMs). As the classical integral equation method for numerical solutions to elliptic boundary value problems, central to the BEM is the reduction of boundary value problems to the equivalent integral equations on the boundary. This boundary reduction has the advantage of diminishing the number of space dimension by one and of the capability to handle problems involving infinite domains. The former leads to an appreciable reduction in the number of algebraic equations generated for the solutions, as well as much simplified data representation. On the other hand, it is well known that elliptic boundary value problems may have equivalent formulations in various forms of boundary integral equations. This provides a great variety of versions for BEMs. However, irrespective of the variants of the BEMs and the particular numerical implementation chosen, there is a common mathematical framework into which all these BEMs may be incorporated. This paper addresses to the fundamental issues of this common mathematical framework and is devoted to the mathematical foundation underlying the BEM techniques.

☆ This paper is based on a plenary lecture entitled *Boundary Element Methods – Past, Present and the Future*, delivered by the author at the First Chilean Workshop on Numerical Analysis of Partial Differential Equations, Universidad de Concepción, Chile, January 13–16, 2004.

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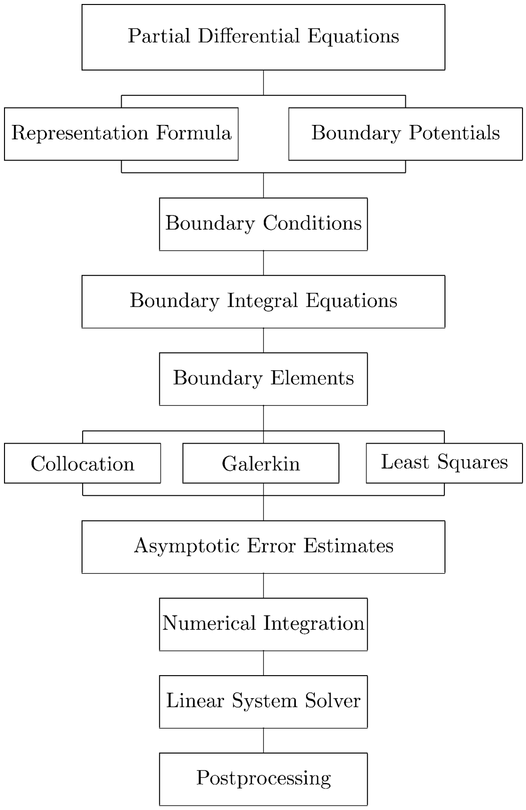

Fig. 1. A flow chart for boundary element methods.

Specifically, this paper will give an expository introduction to the Galerkin-BEM for elliptic boundary value problems from the mathematical point of view. Emphases will be placed upon the variational formulations of the boundary integral equations and the general error estimates for the approximate solutions in appropriate Sobolev spaces. A classification of boundary integral equations will be given based on the Sobolev index. The simple relations between the variational formulations of the boundary integral equations and the corresponding partial differential equations under consideration will be indicated. Basic concepts such as stability, consistency, convergence as well as the condition numbers and ill-posedness will be discussed. Main results obtained by the author and his collaborators over the last 30 years will be summarized. Some numerical experiments will be included to illustrate the fundamental ideas.

BEMs may be considered as application of finite element methods (FEMs) to the boundary integral equations (BIEs) on boundary manifolds. The terminology of BEM originated from the practice of discretizing the boundary manifold of the solution domain for the BIE into boundary elements, resembling the term of finite elements in FEM. As in FEM, the use of the terminology boundary elements in two different contexts; the boundary manifolds are decomposed into boundary elements which are geometric objects, while the boundary elements for approximating solutions of BIEs are actually the finite element functions defined on the boundaries. In fact, the term BEM, nowadays denotes any efficient method for the approximate numerical solution of BIEs. Fig. 1 is a sketch of the general procedure for approximating the solutions of a boundary value problem via the BEMs. As mentioned earlier, we will only concentrate on the Galerkin-BEMs. For the collocation and least-squares BEMs, we refer to the fundamental papers [1,13,30] as well as the review paper [29] on advances on the numerical aspects of the BIE methods.

2. An historic development

In a celebrated paper [7] by Fichera, solutions of the Dirichlet problems for a large class of elliptic equations of higher order with variable coefficients in the plane were obtained by means of the potential of a simple layer. This procedure, which we termed in [14] the *method of Fichera*, leads to singular integral equations of the first kind.